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The General Motion of the Aeroplane

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THE GENERAL MOTION OF THE AEROPLANE

By S. BRODETSKY

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CONTENTS

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INTRODUCTION

1. Comparatively little progress has been made in the mathematical study of the general motion of the aeroplane. Lanchester's phugoids, steady motion, and the small deviations obtained when an aeroplane has an actual motion differing slightly from the steady motion appropriate to the conditions of the controls and engines, represent almost the whole of the mathematics of the rigid dynamics of the aeroplane in the present stage of development of the study of the subject. Interesting results have been obtained by L. Hopf and his collaborators (*Aerodynamik* by L. Hopf, Springer, Berlin, 1934); but general solutions in explicit form are rarely given. Thus the only information of a general character concerning the dynamics of the aeroplane, i.e. unassociated with steady motion, is the theory of Lanchester's phugoids; and although Lanchester published this in 1908, reference to the literature on the subject shows that not only has practically no advance been made on Lanchester's work, but that its significance as a first approximation under certain conditions is not yet fully understood. It is stated by Hopf (*ibid.* p. 231), but he does not specify the exact conditions, and does not study further developments.

It appears, therefore, that there has been almost a generation of stagnation in the mathematical study of aeroplane dynamics, and the object of the present paper is to initiate a systematic mathematical study of the equations of motion of the aeroplane. The general idea is that of finding, in the first instance, an approximate solution, and then improving it by proceeding to a second, or, if necessary, to a still higher approximation. The mathematical process consists of discarding the time as the independent variable, and using one of the Eulerian angles, the pitch, the roll or the yaw, instead—the choice being to a large extent suggested by the nature of the particular kind of motion contemplated; thus the pitching angle must obviously be used when studying longitudinal motions; the yawing angle is convenient in dealing with spinning motions; the rolling angle is convenient in dealing with the sideways roll. Sometimes we can use either one angle or another almost with equal convenience, e.g. in the Immelmann turn.

In order to obtain a process of successive approximations we assume that one component of velocity of the centre of gravity of the machine predominates over the other two components, and that the angular velocity of the machine is small compared to this predominating component; an exact definition of smallness for the angular velocity will be found in the body of the paper. We express all components of velocity of the centre of gravity and the components of angular velocity as ratios of the gliding velocity of the machine, and then consider three typical standard "conditions" of the machine, according as the elevator is adjusted for "standard normal" flight, "standard diving" flight or "standard stalled" flight, as defined below. Using experimental evidence as to the kind of air-resistance forces obtained in these three types of condition of the machine, we find that first approximations can be deduced by means of a judicious comparison of the orders of magnitude of the various terms in the equations of motion.

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The method can be applied to any condition of the machine, i.e. for any given position of the elevator, or indeed of the controls generally. We can also deal with the case of moving controls, and in this paper we indicate how to deal with the elevator moving during the motion, as e.g. in flattening out from a dive, a problem solved recently by a collaborator.

The method can, of course, be used with the machine as a glider, or with the engines and airscrews in action.

We find with comparative ease that, under certain conditions, Lanchester's phugoids represent a first approximation to general longitudinal normal flight.

We also find that, in addition to Lanchester's first approximation, other first approximation paths can be obtained and various new paths are dealt with briefly, e.g. the "extended" phugoids, which are more correct than Lanchester's phugoids. The method can also be applied to "three-dimensional" phugoids which give the Immelmann turn; the slow spin of the stalled aeroplane; the slow roll; etc. In fact, first approximations to the most important aerobatics have already been obtained by the method of this paper. Some have been worked out in detail already; others are now being investigated by the writer and collaborators. Further, a method has been devised for the systematic study of the equations of motion for the deduction of possible first approximations in an *a priori* manner.

Second and higher approximations are not difficult to obtain, when once a first approximation exists. In particular, the looping motion of an aeroplane in normal flight has been worked out by one of the author's research pupils, to a second approximation, by means of a mathematical process which represents a possible alternative to step-by-step integration and has the advantage of being applicable immediately to any initial conditions.

We shall find it convenient to make extensive use of the notation (slightly modified) and data of *Aerodynamic Theory* (edited by W. F. Durand), Division N, "Dynamics of the airplane", by B. Melvill Jones. The reference will be: "Jones, p. ...".

I. LONGITUDINAL MOTION WITHOUT SCREW THRUST

Equations of motion: the coefficients of statical and dynamical stability, κ, τ

2. We consider first the symmetrical aeroplane in longitudinal motion relative to the air, and to begin with we take the case where there is no screw thrust. Let mg be the total weight acting at the centre of gravity G in the vertical plane of longitudinal motion. Let Gx, Gz (fig. 1) be axes fixed in the machine, and let u, w be the velocity components of G along these axes. Let the x axis make an angle θ with the horizontal, in the sense $z \to x$, and let q ($\equiv d\theta/dt$) be the angular velocity of the machine. Let the air resistance produce force components X, Z along the axes Gx, Gz; and let M in the same sense as q be the moment produced by the air resistance.

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It is convenient to define the directions of Gx, Gz relative to the machine as follows. We assume the tail plane and elevator to be in some definite and fixed position in the aeroplane (the rudder and ailerons being of course in their neutral or zero positions), and we choose Gx to be that direction relative to the machine for which, when there is no angular motion, the air-resistance moment is zero. Gz is then taken perpendicular to Gx in the sense shown in fig. 1. If the tail plane or the elevator is turned, this direction $M = 0$ is changed too.

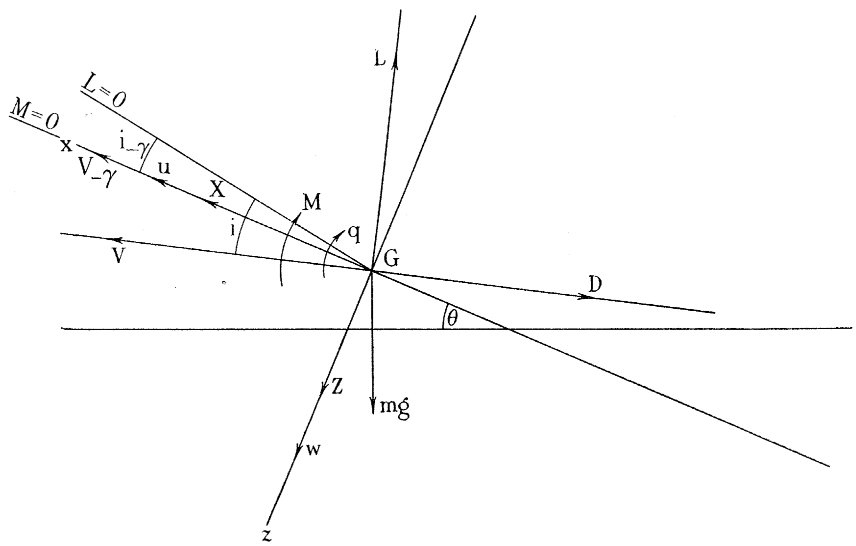

FIG. 1

Let V be the resultant velocity of G, i.e. $\sqrt{(u^2+w^2)}$, and in the case of steady motion let us add a suffix to V or to any other symbol in order to indicate, when necessary, the direction of Gx relative to the horizontal. Let γ be the gliding angle of the machine, with the given and fixed position of the tail plane and elevator. The circumstances of the steady glide are $u = V_{-\gamma}$, $w = 0$, $\theta = -\gamma$, $q = 0$, and the equations of the steady glide are

$$0 = mg\sin\gamma + X_{-\gamma}, \quad 0 = mg\cos\gamma + Z_{-\gamma}, \quad 0 = M_{-\gamma}, \tag{2·1}$$

while the equations of general longitudinal motion are

$$m(\dot{u}+qw) = -mg\sin\theta + X, \quad m(\dot{w}-qu) = mg\cos\theta + Z, \quad B\dot{q} = M, \tag{2·2}$$

where B is the moment of inertia of the machine about an axis at G perpendicular to the longitudinal plane.

By the theory of dimensions, we can, for a given shape of machine, assume that

$$X/\rho V^2 S, \quad Z/\rho V^2 S, \quad M/\rho V^2 Ss,$$

where ρ is the density of the air, S is the wing area, and s is the semi-wing span, are functions of the dimensionless arguments

$$w/V, \quad sq/V,$$

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where s is used as a characteristic length in the machine. As usual, we neglect the viscosity and elasticity of the air; we also neglect the possibility that X, Z may involve accelerations; but, in the case of M, we introduce a term involving the w acceleration (Jones, p. 124) when discussing the stability of steady motion.

If L, D are the lift and drag of the machine, then

$$X = \frac{w}{V}L - \frac{u}{V}D, \quad Z = -\frac{u}{V}L - \frac{w}{V}D, \tag{2·3}$$

where $L/\rho V^2 S$, $D/\rho V^2 S$ are functions of w/V, sq/V. Let us write

$$L = \rho V^2 S k_L\left(\frac{w}{V}, \frac{sq}{V}\right), \quad D = \rho V^2 S k_D\left(\frac{w}{V}, \frac{sq}{V}\right), \quad R \equiv \surd(L^2+D^2) = \rho V^2 S k_R\left(\frac{w}{V}, \frac{sq}{V}\right), \tag{2·4}$$

where k_L, k_D, k_R are the usual "coefficients", expressed as dependent upon both the incidence and the rotation. We at once get

$$X_{-\gamma} = -\rho V_{-\gamma}^2 S k_D(0,0), \quad Z_{-\gamma} = -\rho V_{-\gamma}^2 S k_L(0,0), \tag{2·5}$$

so that

$$\rho V_{-\gamma}^2 S k_D(0,0) = mg\sin\gamma, \quad \rho V_{-\gamma}^2 S k_L(0,0) = mg\cos\gamma, \quad \rho V_{-\gamma}^2 S k_R(0,0) = mg, \tag{2·6}$$

and

$$\tan\gamma = \frac{k_D(0,0)}{k_L(0,0)}. \tag{2·7}$$

The moment M is zero in the steady motion, when $\dot{u}$, w, $\dot{w}$, q are zero. If now these variations from the steady motion are small, M is linear in terms of these quantities. Owing to the choice of the direction Gx we can obviously omit $\dot{u}$; it follows that, using variables of zero dimensions, we can write

$$M = -\textit{a constant times } \rho V^2 S s\left\{\kappa\frac{w}{V} + \frac{sq}{V} + \lambda\frac{s\dot{w}}{V^2} + k_{M2}\left(\frac{w}{V}, \frac{sq}{V}\right)\right\}, \tag{2·8}$$

where k_{M2} contains terms of the second and higher orders in the variables indicated, and κ, λ are constants depending on the aeroplane and the condition of the controls, position of centre of gravity, etc.

Let us now introduce the notation

$$\frac{V}{V_{-\gamma}} \equiv V', \quad \frac{u}{V_{-\gamma}} \equiv u', \quad \frac{w}{V_{-\gamma}} \equiv w', \quad \frac{V_{-\gamma}}{g}q \equiv q', \quad \frac{sg}{V_{-\gamma}^2} \equiv s'; \tag{2·9}$$

the equations of motion (2·2) become

$$\left.\begin{aligned} q'\left(\frac{du'}{d\theta} + w'\right) &= -\sin\theta - \sin\gamma\, V'^2\left\{\frac{u'}{V'}\frac{k_D(w'/V', s'q'/V')}{k_D(0,0)} - \frac{w'}{V'}\cot\gamma\frac{k_L(w'/V', s'q'/V')}{k_L(0,0)}\right\}, \\ q'\left(\frac{dw'}{d\theta} - u'\right) &= \cos\theta - \cos\gamma\, V'^2\left\{\frac{w'}{V'}\tan\gamma\frac{k_D(w'/V', s'q'/V')}{k_D(0,0)} + \frac{u'}{V'}\frac{k_L(w'/V', s'q'/V')}{k_L(0,0)}\right\}, \\ q'\frac{dq'}{d\theta} &= -\tau V'^2\left\{\kappa\frac{w'}{V'} + \frac{s'q'}{V'} + \lambda\frac{s'q'}{V'^2}\frac{dw'}{d\theta} + k_{M2}\left(\frac{w'}{V'}, \frac{s'q'}{V'}\right)\right\}, \end{aligned}\right\} \tag{2·10}$$

in which the independent variable t has been replaced by θ.

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The constant κ will be called the *coefficient of statical stability*, since the machine is statically stable, neutral, or unstable according as κ is positive, zero, or negative.

The constant τ will be called the *coefficient of dynamical stability*, for reasons that will soon be adduced.

The values of κ and τ determine the different types of first approximations deducible by the method of this paper.

Equations of motion: first approximation

3. If we restrict ourselves to motions in which:

V' never becomes large or small, i.e. V' is of zero order in terms of some small quantity to be defined in each condition of the aeroplane;

w' is small, at least of the first order; so that

u' is also of zero order (never becoming small, so that it is always of the same sign which we shall assume to be positive); and

$s'q'$ is small, at least of the first order;

then we can write $V' = \surd(u'^2+w'^2) = u'(1+\frac{1}{2}w'^2/u'^2)$, and the equations of motion become approximately

$$\left.\begin{aligned} q'\left(\frac{du'}{d\theta}+w'\right) &= -\sin\theta-\sin\gamma(u'^2+k_{Dw}u'w'+k_{Dq}u's'q')+\cos\gamma\, u'w', \\ q'\left(\frac{dw'}{d\theta}-u'\right) &= \cos\theta-\cos\gamma(u'^2+k_{Lw}u'w'+k_{Lq}u's'q')-\sin\gamma\, u'w', \\ q'\frac{dq'}{d\theta} &= -\tau(\kappa u'w'+u's'q'), \end{aligned}\right\} \tag{3·1}$$

where

$$\left.\begin{aligned} k_{Lw} &= \frac{1}{k_L(w'/V',\, s'q'/V')}\,\frac{dk_L(w'/V',\, s'q'/V')}{d(w'/V')}, \\ k_{Lq} &= \frac{1}{k_L(w'/V',\, s'q'/V')}\,\frac{dk_L(w'/V',\, s'q'/V')}{d(s'q'/V')}, \end{aligned}\right\} \tag{3·2}$$

in which w'/V', $s'q'/V'$ are put zero after the differentiation; similar definitions hold for k_{Dw}, k_{Dq}.

It is convenient to have also the approximate equations of motion in Jones's notation. We get, after a little manipulation,

$$\left.\begin{aligned} q'\left(\frac{du'}{d\theta}+w'\right) &= -\sin\theta-\sin\gamma\, u'^2-\left(\frac{x_w}{k_R}u'w'+\frac{x_q}{k_R}u's'q'\right), \\ q'\left(\frac{dw'}{d\theta}-u'\right) &= \cos\theta-\cos\gamma\, u'^2-\left(\frac{z_w}{k_R}u'w'+\frac{z_q}{k_R}u's'q'\right), \\ q'\frac{dq'}{d\theta} &= -\tau(\kappa u'w'+u's'q'), \end{aligned}\right\} \tag{3·3}$$

in which x_w, z_w, ... are Glauert's non-dimensional derivatives: x_u, z_u do not occur explicitly, being absorbed in u'^2. Further, x_w, z_w, m_q as given by Jones remain un-

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changed; but we shall assume that x_q, z_q as given by Jones have been multiplied by c/s, and m_w as given by Jones has been multiplied by s/c, since he uses the chord c instead of the semi-span s as the characteristic length for the machine.

We note that

$$k_{Lw} = \frac{z_w}{k_L} - \tan\gamma, \quad k_{Dw} = \frac{x_w}{k_D} + \cot\gamma, \quad k_{Lq} = \frac{z_q}{k_L}, \quad k_{Dq} = \frac{x_q}{k_D}. \tag{3·4}$$

In what follows we shall find k_{Lw}, k_{Dw} directly from the definitions (4·1) below; but for k_{Lq}, k_{Dq} we shall take their values in terms of Jones's derivatives, since the functional forms $k_L(w'/V', s'q'/V')$, $k_D(w'/V', s'q'/V')$ are known only when there is no rotation, i.e. $q' = 0$.

The three standard conditions of the symmetrical aeroplane

4. We shall in this first attack on the general problem consider three different standard conditions of the machine, produced by different adjustments of the tail plane and elevator:

(I) *Standard normal condition*,
(II) *Standard diving condition*, and
(III) *Standard stalled condition*.

In order to define these standard conditions, let us measure the angle of true incidence i from that direction of motion relative to the machine for which the lift is zero (Fig. 1). If the tail plane or elevator is turned, this direction $L = 0$ is also changed slightly, and we measure i from the direction $L = 0$ appropriate to the actual position of tail plane and elevator. In the absence of rotation, we have the lift and drag coefficients k_L and k_D as functions of i. Hence, for any condition of the machine, which of course defines the incidence of the glide, we have the gliding angle γ as a function of the angle of gliding incidence $i_{-\gamma}$. When this angle is zero, k_L is zero, so that the gliding angle is $\frac{1}{2}\pi$, and the dive is vertical.

Hence we define (II), the *standard diving condition*, as being the condition of the machine *when $i_{-\gamma}$ is zero*, so that $\gamma = \frac{1}{2}\pi$. This definition is obvious.

We define (III), the *standard stalled condition*, as being the condition of the machine when the incidence in the steady glide is that which gives maximum lift. This is also an obvious definition.

To define (I), the *standard normal condition*, we note that at cruising incidence the gliding angle for a conventional aeroplane is more or less the same as the angle of incidence measured from the direction of zero lift of the machine. We therefore define the standard normal condition as being the condition of the machine for which $\gamma = i_{-\gamma}$, *both being small* (we add the last condition, since we can have $\gamma = i$ for a large angle of incidence, beyond stalling incidence). The ideal aeroplane discussed by Bryan (*Stability in Aviation*, ch. v) belongs to this type, if his α is small, but not too small, say between

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3° and 10°. In the steady glide of an aeroplane in standard normal condition the direction $L = 0$ is horizontal.

For a conventional machine we can write, approximately, in the absence of rotation,

$$k_L(w/V, 0) \equiv k_L(i) = k_{LS} \sin ni, \quad k_D(w/V, 0) \equiv k_D(i) = k_{D0} + k_{D1} \sin^2 i, \tag{4·1}$$

where $i = i_{-\gamma} + \sin^{-1}(w/V)$, and n is such a number that the stalling incidence is $\pi/2n$ (n is generally between 4 and 6); k_{LS} is the maximum value of k_L, namely, at the stalling incidence; k_{D0} is the minimum value of k_D, and k_{D1} is another constant, whose value (see Hopf, *ibid.* p. 99, fig. 76) is of the same order as k_{LS}.

The definitions (4·1) take into account the stall, but are not very accurate for small i, when it is better to use $k_L(i) = ai$, $k_D(i) = b + ci^2$ where a, b, c are certain constants. Orders of magnitude are not affected, however, if we use (4·1) even for small i.

k_{LS} is usually a number like 0·6; k_{D0} is usually a number like 0·02. We do not need the actual values here, but only some indication of the orders of magnitude as they occur in practice.

Our value of k_D makes it a minimum at $i = 0$. This is not quite correct, but it is easily seen that this is of no serious consequence in what follows.

We can write (2·6), (2·7) in the form

$$\rho V_{-\gamma}^2 S k_D(i_{-\gamma}) = mg \sin\gamma, \quad \rho V_{-\gamma}^2 S k_L(i_{-\gamma}) = mg \cos\gamma, \quad \rho V_{-\gamma}^2 S k_R(i_{-\gamma}) = mg, \tag{4·2}$$

$$\tan\gamma = \frac{k_D(i_{-\gamma})}{k_L(i_{-\gamma})}. \tag{4·3}$$

In order to calculate k_{Lw} and k_{Dw}, we can, in the definitions (3·2), make $s'q'/V'$ zero before we differentiate with respect to w'/V'. Hence

$$\left.\begin{aligned} k_{Lw} &= \operatorname*{Lt}_{w'/V'=0} \frac{k_L(i_{-\gamma} + \sin^{-1}(w'/V'), 0) - k_L(i_{-\gamma}, 0)}{(w'/V')\, k_L(i_{-\gamma}, 0)} = \left\{\frac{1}{k_L(i)}\frac{dk_L(i)}{di}\right\}_{i=i_{-\gamma}} = n \cot ni_{-\gamma}, \\ k_{Dw} &= \operatorname*{Lt}_{w'/V'=0} \frac{k_D(i_{-\gamma} + \sin^{-1}(w'/V'), 0) - k_D(i_{-\gamma}, 0)}{(w'/V')\, k_D(i_{-\gamma}, 0)} = \left\{\frac{1}{k_D(i)}\frac{dk_D(i)}{di}\right\}_{i=i_{-\gamma}} \\ &= \frac{k_{D1}}{k_{LS}} \frac{\sin 2i_{-\gamma}}{\sin ni_{-\gamma}} \cot\gamma, \end{aligned}\right\} \tag{4·4}$$

approximately by (4·1) using (4·3) for $\tan\gamma$.

To find the orders of magnitude of k_{Lq}, k_{Dq}, we use the last two equations of (3·4) with the numerical data given by Jones. We find that practically always k_{Dq} can be safely ignored.

For the three standard conditions we can therefore use:

(I) *Standard normal condition:*

$$k_{Lw} \doteqdot \frac{1}{\sin\gamma}; \quad k_{Dw} \doteqdot \frac{2}{n \sin\gamma}; \quad k_{Lq} \text{ is a number like } 2; \quad k_{Dq} \doteqdot 0.$$

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(II) *Standard diving condition:*

k_{Lw} is infinite. In fact, however, k_{Lw} always occurs with the factor $\cos\gamma$ which is zero. Now

$$\cos\gamma\, k_{Lw} = \sin\gamma \cot\gamma\, n \cot ni_{-\gamma} = \sin\gamma \frac{k_L(i_{-\gamma})}{k_D(i_{-\gamma})} n \cot ni_{-\gamma} = n\sin\gamma \frac{k_{LS}\cos ni_{-\gamma}}{k_{D0}+k_{D1}\sin^2 i_{-\gamma}},$$

and as $i_{-\gamma}\to 0$ so that $\gamma\to\frac{1}{2}\pi$, this quantity tends to the limit nk_{LS}/k_{D0}, a number like 80 or 100 for any conventional aeroplane.

k_{Dw} is, on our hypothesis, zero at $i_{-\gamma}=0$; it is in fact quite negligible for an actual machine.

k_{Lq}, like k_{Lw}, is infinite; but, like k_{Lw}, it occurs only in combination with $\cos\gamma$. Now $\cos\gamma\, k_{Lq}$ can be written z_q/k_D, and this is a number like 15 or 20 at standard diving incidence.

k_{Dq} is ignored.

(III) *Standard stalled condition:*

$$k_{Lw}=0; \quad k_{Dw} \doteqdot \frac{\sin\pi/n}{\sin\gamma}, \text{ about } 1\tfrac{1}{2} \text{ or } 2; \quad k_{Lq} \text{ is very small}; \quad k_{Dq}\doteqdot 0.$$

We now proceed to consider the numerical values of κ, λ, τ, s' for ordinary aeroplanes.

If we write M as far as the first powers of q, w, $\dot{w}$, we have, with Jones's notation (p. 133) adjusted for s instead of c,

$$M = -\frac{B}{ms^2}(\rho VSsm_w w + \rho Ss^2 m_{\dot{w}}\dot{w} + \rho VSs^2 m_q q).$$

Hence, by (2·8), we have
$$\kappa = \frac{m_w}{m_q}, \quad \lambda = \frac{m_{\dot{w}}}{m_q}, \tag{4·5}$$

and
$$M = -\frac{B}{ms}\rho V^2 Sm_q\left(\kappa\frac{w}{V}+\frac{sq}{V}+\lambda\frac{s\dot{w}}{V^2}\right).$$

If we take θ as the independent variable and put $(V_{-\gamma}/g)\, q \equiv q'$ in (2·10), we find

$$q'\frac{dq'}{d\theta} = -\frac{\rho V_{-\gamma}^2 S}{ms}\left(\frac{V_{-\gamma}}{g}\right)^2 m_q V'^2\left(\kappa\frac{w'}{V'}+\frac{s'q'}{V'}+\lambda\frac{s'q'}{V'^2}\frac{dw'}{d\theta}\right),$$

so that we have
$$\tau = \frac{\rho V_{-\gamma}^2 S}{ms}\left(\frac{V_{-\gamma}}{g}\right)^2 m_q. \tag{4·6}$$

But, by (4·2), we can write $\rho V_{-\gamma}^2 Sk_R = mg$,

where k_R is the resultant of the lift and drag coefficients for the incidence of the steady glide.

Hence
$$\tau = \frac{m}{\rho Ss}\frac{m_q}{k_R^2} = \frac{\mu m_q}{k_R^2}, \tag{4·7}$$

where μ is Glauert's dimensionless parameter as defined by Jones (p. 183).

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For any ordinary aeroplane m_q, and therefore τ, is always positive.

In conventionally shaped aeroplanes we find that τ_D, the value of τ in the standard diving condition, τ_N, the value of τ in the standard normal condition, and τ_S, the value of τ in the standard stalled condition, bear to one another ratios like

$$\tau_D : \tau_N : \tau_S = 4000 : 40 : 1, \tag{4·8}$$

although there are, of course, considerable deviations from these average ratios in actual machines.

For ordinary flying altitudes, μ is in general a number of the order 10, and increases with height above the earth's surface, e.g. in stratospheric flight; k_R is a number like 0·02 in the standard diving condition, 0·2 in the standard normal condition, and 0·6 in the standard stalled condition. Hence we see that for ordinary aeroplanes τ_N is several hundreds, so that τ_D is of the order of many thousands, while τ_S can be generally assumed to be of the order 10.

Although τ decreases rapidly from τ_D to τ_N and then to τ_S, it appears that beyond stalling incidence the dynamical stability coefficient increases again, for in the expression

$$\tau = \frac{\mu m_q}{k_R^2},$$

k_R decreases slightly after stalling incidence, and m_q increases after stalling incidence.

It is not possible to define κ numerically in any general manner, since its value depends upon the position of the centre of gravity. The published results of wind tunnel and other observations do not at present afford much information about κ, and it would seem to be desirable that experimenters should give the value of κ in any motion investigated. As far as such information is available we can state that in the standard stalled condition κ is certainly positive, and it can perhaps be assumed to be a number like 6 or 8. If we assume stability, then for both the standard normal and the standard diving conditions of the machine we may take κ positive but small; although it may be as much as $\frac{1}{3}$ or $\frac{1}{2}$, and in extreme cases can even be a number like unity at the former. But we have to be prepared to use both positive and negative values of κ in normal and diving flight, so long as they are not large numerically.

λ is positive and varies between 0·5 and zero.

The value of s' is rather uncertain, since

$$s' \equiv \frac{gs}{V_{-\gamma}^2},$$

and, of course, the semi-span s and the gliding velocity $V_{-\gamma}$ differ widely from machine to machine. If the machine, as used by Jones, is small, with s about 15 or 20 ft., then, with a stalling velocity like 65 miles per hour, s' is a number between 1/15 and 1/20 in the standard stalled condition; between 1/50 and 1/100 in the standard normal condition; and between 1/300 and 1/600 in the standard diving condition.

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But if the machine is a large one, with s about 40 or 50 ft., or even more, then the value of s' is correspondingly larger, unless $V_{-\gamma}$ is also larger, which is usually the case.

In this paper we shall use approximations adapted to the machine used by Jones.

It is useful to note that
$$\tau s' = \frac{m_q}{k_R}, \tag{4·9}$$
and that, in a conventional aeroplane, the values of $\tau s'$ at standard diving, normal and stalled conditions are numbers like 100, 10, and 2/3 respectively.

Longitudinal stability

Usual value of τ in standard normal condition

5. The discarding of t as the independent variable is fundamental to the method of this paper. In order, however, to obtain clear notions about the coefficient of dynamical stability, τ, it is necessary to consider the stability problem with t as independent variable. For this purpose we can use equations (3·1) with the definition
$$q' \equiv D\theta,$$
where D means differentiation with respect to the time, multiplied by $V_{-\gamma}/g$; so that
$$q' \frac{d}{d\theta} = D \equiv \frac{V_{-\gamma}}{g} \frac{d}{dt}. \tag{5·1}$$

The steady glide is defined by
$$u' = 1, \quad w' = 0, \quad \theta = -\gamma, \quad q' = 0;$$
and for the stability discussion we use
$$u' = 1 + u_1, \quad w' = w_1, \quad \theta = -\gamma + \theta_1, \quad q' = D\theta_1,$$
where quantities with suffix 1 are infinitesimal. To the first order of these infinitesimal quantities, the equations of motion (3·1) become
$$\left.\begin{aligned} Du_1 &= \sin\gamma - \theta_1 \cos\gamma - \sin\gamma (1 + 2u_1 + k_{Dw} w_1 + k_{Dq} s' D\theta_1) + \cos\gamma\, w_1, \\ Dw_1 - D\theta_1 &= \cos\gamma + \theta_1 \sin\gamma - \cos\gamma (1 + 2u_1 + k_{Lw} w_1 + k_{Lq} s' D\theta_1) - \sin\gamma\, w_1, \\ D^2\theta_1 &= -\tau(\kappa w_1 + s' D\theta_1 + \lambda s' Dw_1); \end{aligned}\right\} \tag{5·2}$$
where k_{Lw}, k_{Dw}, k_{Lq}, k_{Dq} are as defined in (3·2).

The determinantal biquadratic for deciding stability is therefore
$$\begin{vmatrix} D + 2\sin\gamma, & \sin\gamma\, k_{Dw} - \cos\gamma, & s' \sin\gamma\, k_{Dq} D + \cos\gamma, \\ 2\cos\gamma, & D + \sin\gamma + \cos\gamma\, k_{Lw}, & \overline{-1 + s' \cos\gamma\, k_{Lq}} D - \sin\gamma, \\ 0, & \lambda s' \tau D + \kappa\tau, & D^2 + s' \tau D. \end{vmatrix} \tag{5·3}$$

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The coefficients of the various powers of D in this biquadratic are

$$\left.\begin{array}{l} 1, \\ s'\tau+3\sin\gamma+\cos\gamma\, k_{Lw}+\lambda s'\tau(1-s'\cos\gamma k_{Lq}), \\ \kappa\tau+s'\tau(3\sin\gamma+\cos\gamma\,\overline{k_{Lw}-\kappa k_{Lq}})+\lambda s'\tau(3\sin\gamma-2s'\sin\gamma\cos\gamma\,\overline{k_{Lq}-k_{Dq}}) \\ \qquad\qquad +2+2\sin\gamma\cos\gamma(k_{Lw}-k_{Dw}), \\ \tau(3\kappa\sin\gamma+2\lambda s'+2s'\sin\gamma\cos\gamma\,\overline{k_{Lw}-k_{Dw}-\kappa k_{Lq}+\kappa k_{Dq}}+2s'), \\ 2\kappa\tau. \end{array}\right\} \qquad (5\cdot4)$$

The conditions of stability are that these coefficients must all be positive, and, in addition, the Routh discriminant must be positive. We see that a necessary condition of stability is that $\kappa\tau$ shall be positive. Since τ is always positive for a conventionally shaped machine, we must have κ positive—this is indeed the important condition for stability of the steady glide.

If we use the approximate values of k_{Lw}, k_{Dw}, k_{Lq}, k_{Dq} of § 4, and the orders of magnitude of s', we find that, in each of the three standard conditions of a conventional aeroplane, the coefficients (5·4) are all positive if κ is positive without being absurdly large. Routh's discriminant is therefore the only additional condition, and it is easy to prove that:

(I) In the standard normal condition we have longitudinal stability of the steady glide if κ is positive, and if in addition τ_N is greater than a quantity of order -2 in terms of $\sin\gamma$, *which is small in normal condition.*

(II) In the standard diving condition, κ positive is sufficient to ensure stability of the steady glide.

(III) In the standard stalled condition, the longitudinal stability of the steady glide can only be ensured if τ_N is at least as big as of order -2 in terms of the gliding angle in the normal condition.

We shall therefore assume all through this paper, that τ_N is at least of order -2 in terms of the small quantity $\sin\gamma$ in the normal condition, and that if necessary it may be assumed to be of order -3 in this small quantity.

(I) Standard normal condition: three subtypes of phugoids

6. Let us now consider the general motion when the aeroplane is in standard normal condition. In this condition of the machine, $\sin\gamma$ is small, about 1/7 say. We define $\sin\gamma$ as the first order small quantity; then using Jones's machine as suggested in § 4, we see that s' is of the second order. We can also take k_{Lw} and k_{Dw} to be of order -1, k_{Lq} of order zero, and k_{Dq} negligible. As for τ_N, we can take it to be of order -2 in $\sin\gamma$, and, if we like, of order -3.

We shall examine three subtypes which yield comparatively simple first approximations:

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(*a*) κ of *zero* order; τ_N of order -2, or larger;
(*b*) κ *small*, of the *first* order; τ_N of order -3, or larger;
(*c*) κ *negligible* (machine statically neutral); τ_N of order -2, or larger.

For each subtype we shall find a first approximation: (*a*) gives Lanchester's phugoids; (*b*) and (*c*) give new first approximation paths.

7. (*a*) *κ of zero order; τ_N of order -2, or larger; Lanchester's phugoids.* Let us assume that q is small, and of such a size that q' is of zero order. Then the third equation of motion (3·1) gives

$$\kappa w' + s'q' = -\frac{1}{\tau u'} q' \frac{dq'}{d\theta}. \tag{7·1}$$

Since τ_N is at least as big as of order -2, it follows that the right-hand side of (7·1) is at least of the second order of smallness; and the left-hand side must also be of this order of smallness. But $s'q'$ is of the second order; hence, since κ is of zero order, it follows that w' must be a small quantity of the second order.

If now we omit in the first two equations (3·1) all quantities which are of the same order as $\sin\gamma$, or smaller, we get the equations

$$q'\frac{du'}{d\theta} = -\sin\theta, \quad -q'u' = \cos\theta - u'^2, \quad w' = -\frac{s'}{\kappa} q' - \frac{1}{\kappa\tau u'} q' \frac{dq'}{d\theta}, \tag{7·2}$$

as a first approximation. The first two equations give u', q', and then w' is given by the third equation.

In terms of the primitive equations of motion (2·2) this means that we assume for the components of air thrust X, Z,

$$\text{(i)}\ X = 0, \quad \text{(ii)}\ Z = -\rho S k_L u^2. \tag{7·3}$$

Thus (i) *we neglect the drag*. We are at present taking the air-screws as not in action; we shall see later, §16, that in practice, generally, the action of the air-screws is irrelevant to the validity of Lanchester's phugoids as a first approximation, our present assumption being replaced by the assumption that both the drag and the air-screw thrust can be neglected (see Hopf, *ibid.* p. 231); and

(ii) *we take the lift as a constant times the square of the forward velocity*, which means in effect that we assume constant angle of incidence, and neglect the effect of changes in the w velocity component; further

(iii) *we assume that longitudinally the machine adjusts itself instantaneously*, so that, to our degree of approximation, the motion is along the axis Gx, w' being of the second order. Lanchester describes this as meaning that *the moment of inertia of the machine is neglected*; this is a rough and ready description of what actually happens if τ_N is of order -2, or larger.

Eliminating q' between the first two equations (7·2), we deduce

$$\cos\theta\frac{du'}{d\theta} - u'^2\frac{du'}{d\theta} = u'\sin\theta, \tag{7·4}$$

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so that $$u' \cos\theta = \tfrac{1}{3}u'^3 + A, \tag{7·5}$$

where A is an arbitrary constant. This is the velocity equation for Lanchester's phugoids, which are thus obtained as a first approximation of the equations of motion of an aeroplane in standard normal condition with κ of zero order.

If we write (7·5) in the form

$$u'^3 - 3u' \cos\theta = n^3 - 3n, \tag{7·6}$$

where n is positive and equal to the value of u' when $\theta = 0$, we get

$$\cos\theta = \frac{n^2}{3}\left(\frac{u'}{n}\right)^2 - \left(\frac{n^2}{3} - 1\right)\left(\frac{u'}{n}\right)^{-1}. \tag{7·7}$$

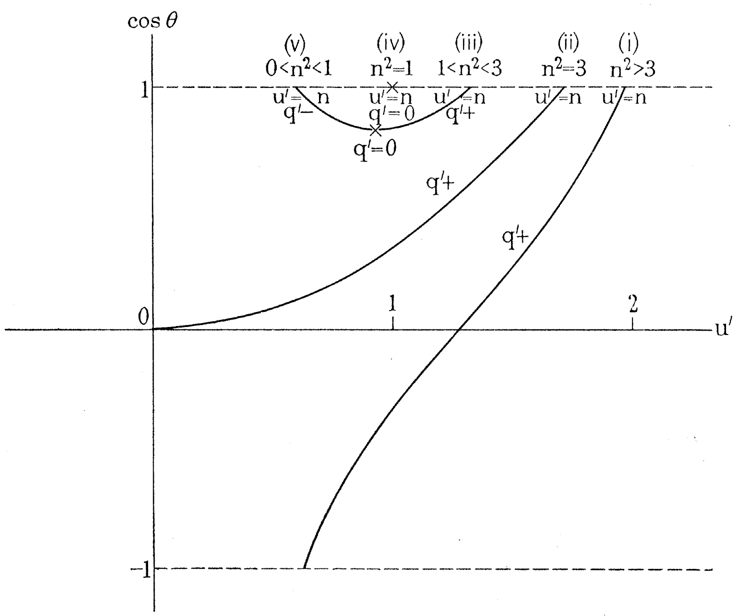

FIG. 2

We deduce from the first equation (7·2) that

$$q' = \frac{d(\cos\theta)}{du'}, \tag{7·8}$$

so that (7·6) gives $$\frac{q'}{n} = \frac{2}{3}\left(\frac{u'}{n}\right) + \left(\frac{1}{3} - \frac{1}{n^2}\right)\left(\frac{u'}{n}\right)^{-2}. \tag{7·9}$$

The radius of curvature ρ is, approximately, $u/\dot{\theta}$, i.e. $(V^2_{-\gamma}/g)\, u'/q'$; hence

$$\rho = \frac{V^2_{-\gamma}}{g}\left\{\frac{2}{3} + \left(\frac{1}{3} - \frac{1}{n^2}\right)\left(\frac{u'}{n}\right)^{-3}\right\}^{-1}. \tag{7·10}$$

It is not necessary to consider here the detailed properties of Lanchester's phugoids; but in order to prepare the ground for other first approximations, it is useful to examine briefly the main forms of these phugoids as defined by the constants in the relation between $\cos\theta$ and u'.

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If we plot $\cos\theta$ against u' from (7·7), we obtain the various cases of fig. 2, and the following classification of the paths:

(i) $n^2 > 3$; tumbler or "looping" phugoids;
(ii) $n^2 = 3$; semicircular phugoid;
(iii) $1 < n^2 < 3$; inflected or "undulating" phugoids, really identical with (v), but starting at a lowest point;
(iv) $n^2 = 1$; straight-line phugoid;
(v) $0 < n^2 < 1$; inflected or "undulating" phugoids, really identical with (iii), but starting at a highest point.

Lanchester's phugoids are obtained where κ is of zero order, whether positive or negative, i.e. if the aeroplane is excessively stable, or excessively unstable, statically. This is rather rare in practice. To a first approximation the two cases give the same paths: the difference between the two cases is now being studied.

8. *(b) κ small, of the first order; τ_N of order -3, or larger; extended phugoids.* If κ is small, of the same order as $\sin\gamma$, we do not obtain the simplified forms of the first two equations (3·1) that lead to the Lanchester phugoids, because the third equation, which makes $\kappa w'$ of the second order, now makes w' only of the first order. If, however, we take τ_N to be of order -3 in $\sin\gamma$, then we can ignore the right-hand side in (7·1) altogether.

The three equations of motion (3·1) are now, to a first approximation,

$$q'\frac{du'}{d\theta} = -\sin\theta, \quad -q'u' = \cos\theta - u'^2 - k_{Lw}u'w', \quad \kappa w' + s'q' = 0. \tag{8·1}$$

Our approximations in terms of the air-thrust components are

$$\text{(i)}\ \ X = 0, \quad \text{(ii)}\ \ Z = -\rho S k_L u^2\left(1 + k_{Lw}\frac{w}{u}\right). \tag{8·2}$$

Hence *we are once again neglecting the drag*. But *in the* case of the *lift*, the *effect of changes in the w velocity component is included*, and to this extent the approximation is better than in the case of Lanchester's phugoids. *The moment of inertia is "neglected"*.

The case κ small, whether positive or negative, is important in practice, and since the assumption that τ_N is of order -3 is not really a serious restriction, it would appear that the "extended" phugoids now to be discussed are of better validity than Lanchester's phugoids. They also represent a less violent approximation, as just shown.

Substituting for w' from the third equation (8·1) in the second, and putting

$$\frac{\kappa}{s'k_{Lw}} \equiv \frac{K}{1-K}, \tag{8·3}$$

we get

$$q'\frac{du'}{d\theta} = -\sin\theta, \quad -\frac{q'}{K}u' = \cos\theta - u'^2, \tag{8·4}$$

giving

$$\cos\theta\frac{du'}{d\theta} - u'^2\frac{du'}{d\theta} = \frac{u'}{K}\sin\theta, \tag{8·5}$$

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so that
$$\cos\theta\, u'^{K} - \frac{K}{K+2} u'^{K+2} = A, \tag{8·6}$$
where A is an arbitrary constant.

Let n be the value of u' at $\theta = 0$, when $\cos\theta = 1$; then
$$A = n^{K} - \frac{K}{K+2} n^{K+2},$$
and we easily deduce the equation
$$\cos\theta = \frac{n^2 K}{K+2}\left(\frac{u'}{n}\right)^2 - \left(\frac{n^2 K}{K+2} - 1\right)\left(\frac{u'}{n}\right)^{-K}; \tag{8·7}$$
and, since $q' = d(\cos\theta)/du'$, we get
$$\frac{q'}{n} = \frac{2K}{K+2}\frac{u'}{n} + K\left(\frac{K}{K+2} - \frac{1}{n^2}\right)\left(\frac{u'}{n}\right)^{-K-1}. \tag{8·8}$$
The radius of curvature of the path is given by
$$\rho = \frac{V_{-\gamma}^2}{g}\left\{\frac{2K}{K+2} + K\left(\frac{K}{K+2} - \frac{1}{n^2}\right)\left(\frac{u'}{n}\right)^{-K-2}\right\}^{-1}. \tag{8·9}$$

If we compare these results with those of § 7, we see that we have found extended phugoids, more general than Lanchester's. If κ is of zero order, whether positive or negative, $\kappa/s'k_{Lw}$ is large, so that $K \to 1$, and we get Lanchester's phugoids.

We shall not discuss the extended phugoids in any detail here; they are being studied with a collaborator. We shall indicate, however, the main forms that they assume under different conditions.

In normal condition k_{Lw} is positive. If κ is positive, K is positive and less than unity. If κ is negative, and $\kappa/s'k_{Lw}$ is numerically greater than 1, K is positive and greater than unity. In either case we get $\cos\theta$ in terms of u' analogous to fig. 2, with the nature of the paths as follows:

(i) $n^2 > (K+2)/K$; looping;
(ii) $n^2 = (K+2)/K$; semicircular;
(iii) $1 < n^2 < (K+2)/K$; undulating;
(iv) $n^2 = 1$; straight line;
(v) $0 < n^2 < 1$; undulating.

The paths (iii) and (v) are really the same paths with different starting points.

If κ is negative and $\kappa/s'k_{Lw}$ is numerically less than unity, K is negative. Put $L = -K$.

When $L > 2$, $\cos\theta$ in terms of u' is for some of the paths given by fig. 3, and the nature of the paths is as follows:

(i) $0 < n^2 < (L-2)/L$; cusped (analogous to the semicircular phugoid);
(ii) $n^2 = (L-2)/L$; semicircular;
(iii) $(L-2)/L < n^2 < 1$; cusped, or looping;
(iv) $n^2 = 1$; cusped, or straight line, or looping;
(v) $n^2 > 1$; cusped, or looping.

The paths (iii) and (v) are really the same paths with different starting points.

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The loops in these paths differ from those with K positive in that they are described with negative rotation: these looped paths are like Lanchester's looping phugoids but inverted.

There are also paths entirely different in character: for these n does not exist at all. When $L \leqslant 2$ we obtain similar results.

The special case $K \to \infty$, or $\kappa = -s' k_{Lw}$, is interesting. Equations (8·4) now reduce to

$$q' \frac{du'}{d\theta} = -\sin\theta, \quad u'^2 = \cos\theta, \tag{8·10}$$

and we always get either a straight line, or a semicircular path. The fact that we get, for this value of K, singular solutions is associated with the range of validity of the approximation for any value of K.

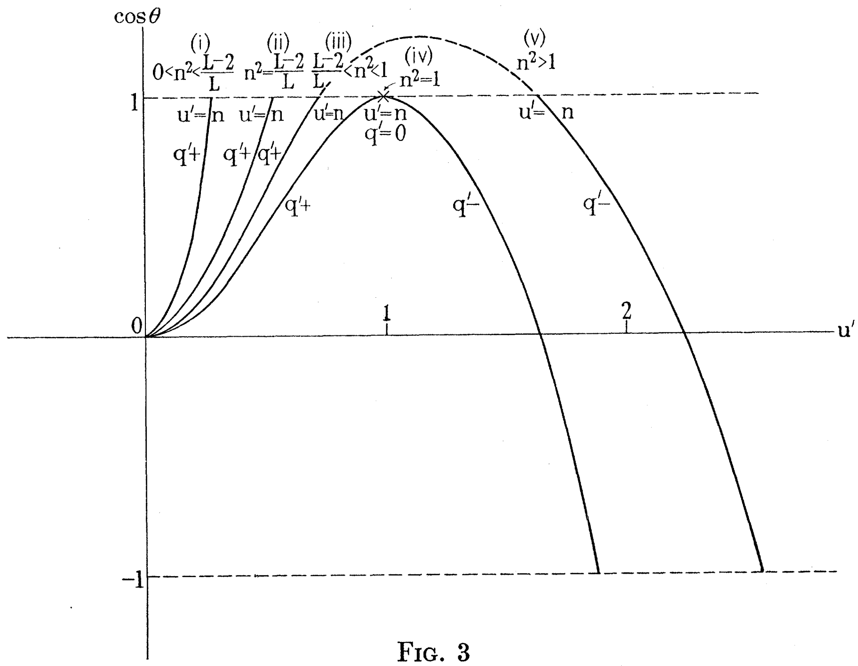

FIG. 3

9. (*c*) *κ negligible, τ_N of order -2 or larger; neutral phugoids.* If κ is negligible, the term $\kappa w'$ in the third equation of motion (3·1) can be discarded in comparison with the term $s'q'$. We are thus left with the approximate equation

$$q' \frac{dq'}{d\theta} = -\tau s' u' q', \tag{9·1}$$

and since, by (4·9), $\tau s'$ is of order -1 in normal condition, we get, in general, $q' = 0$ as a first approximation.

Let w' be of the first order. Then we get, as a first approximation of the equations (3·1),

$$q' \frac{du'}{d\theta} = -\sin\theta, \quad -q'u' = \cos\theta - u'^2 - k_{Lw} u' w', \quad q' \frac{dq'}{d\theta} = -\tau s' u' q'. \tag{9·2}$$

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This approximation means

$$\text{(i)}\ \ X=0,\quad \text{(ii)}\ \ Z=-\rho S k_L u^2\left(1+k_{Lw}\frac{w}{u}\right). \tag{9·3}$$

Thus *we are* once more *neglecting the drag*, and *in the lift we are* again *including the effect of changes in the w velocity component*. But in addition, *the moment of inertia is not neglected*, and indeed no assumption is made about the orientation of the machine in the longitudinal plane.

(i) If q', or $d\theta/dt$, is zero initially, then, since (9·1) means $d^2\theta/dt^2 \propto u'\,d\theta/dt$, it follows that $d\theta/dt$ is zero permanently. This means that the machine has no angular motion at all, and Gx points in the constant direction θ.

Using (5·1) the first equation (9·2) now gives, approximately,

$$\frac{V_{-\gamma}}{g}\frac{du'}{dt}=-\sin\theta, \tag{9·4}$$

in which θ is constant, while the second equation becomes

$$w'=\frac{\cos\theta-u'^2}{u'k_{Lw}}. \tag{9·5}$$

Since k_{Lw} is of order -1 this makes w' small and of order 1, as postulated. If θ is zero, we get u' constant, n say, with w' also constant. The path is thus approximately a straight line inclined below the horizon by the small angle $(1-n^2)/n^2k_{Lw}$, and described with uniform speed n.

If, in addition to θ being zero, we also have $n=1$, this line is horizontal; we get an analogue of the straight-line phugoid.

If θ is not zero, we find readily that

$$u'=u'_0-\frac{g\sin\theta}{V_{-\gamma}}t,\quad k_{Lw}w'=\frac{\cos\theta}{u'_0-(g\sin\theta/V_{-\gamma})\,t}-\left(u'_0-\frac{g\sin\theta}{V_{-\gamma}}t\right), \tag{9·6}$$

and the position of the centre of gravity G after time t is given, relative to the initial axes Gx, Gz, by the displacement

$$x=u_0t-\tfrac{1}{2}g\sin\theta\,t^2,\quad k_{Lw}z=-u_0t+\tfrac{1}{2}g\sin\theta\,t^2-\frac{V_{-\gamma}^2}{g}\cot\theta\log_e\left(1-\frac{g\sin\theta}{u_0}t\right), \tag{9·7}$$

in which u_0 is the actual velocity along Gx at $t=0$, and we proceed only so long as $(g\sin\theta/u_0)\,t$ is distinctly less than unity.

For different values of θ and u_0 we get a variety of paths which can be easily plotted; we may call them "neutral" phugoids, since they are distinguished by the fact that the machine is statically neutral. The statically neutral aeroplane is dealt with by R. Fuchs and L. Hopf (*ibid.* pp. 235–9), but no explicit general solution is given.

(ii) If $d\theta/dt$ is not zero initially, so that θ varies during the motion, (9·2) gives

$$q'\frac{du'}{d\theta}=-\sin\theta,\quad -q'u'=\cos\theta-u'^2-k_{Lw}u'w',\quad \frac{dq'}{d\theta}=-\tau s'u'. \tag{9·8}$$

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The first and third equations (9·8) yield for q' the equation

$$q'\frac{d^2q'}{d\theta^2} = \tau s' \sin\theta, \tag{9·9}$$

which can hold so long as θ is small. This equation is now being investigated. When q' is found, (9·8) give u' and w' without any trouble.

Second approximation

(I) (*a*) *Lanchester's phugoids: the loop*

10. When a first approximation has been obtained, it is a comparatively simple matter to deduce more accurate results by proceeding to a second approximation, and then, if necessary, to a third. A second approximation is usually sufficient.

The method is obvious enough. We shall illustrate it by reference to Lanchester's phugoids.

Let u', w', q' be supposed expanded in powers of γ. For Lanchester's phugoids, § 7, we have u' of zero order, w' of the second order, and q' of zero order. Let us then write

$$u' = u_0+\gamma u_1+\gamma^2 u_2+\ldots, \quad w' = \gamma^2 w_2+\gamma^3 w_3+\ldots, \quad q' = q_0+\gamma q_1+\gamma^2 q_2+\ldots, \tag{10·1}$$

where u_0, u_1, ...; w_2, w_3, ...; q_0, q_1, ..., are all quantities of zero order. The notation is obvious.

Substitute in the equations of motion (3·1); we obtain

$$\left.\begin{aligned}
&(q_0+\gamma q_1)\left(\frac{du_0}{d\theta}+\gamma\frac{du_1}{d\theta}\right) = -\sin\theta-\gamma u_0^2,\\
&(q_0+\gamma q_1)\,(-u_0-\gamma u_1) = \cos\theta-(u_0^2+2\gamma u_0 u_1)-\gamma^2 k_{Lw} u_0 w_2,\\
&(q_0+\gamma q_1)\left(\frac{dq_0}{d\theta}+\gamma\frac{dq_1}{d\theta}\right) = -\tau\{\kappa(u_0+\gamma u_1)\,(\gamma^2 w_2+\gamma^3 w_3)+s'(u_0+\gamma u_1)\,(q_0+\gamma q_1)\},
\end{aligned}\right\} \tag{10·2}$$

as far as γ, since k_{Lw}, k_{Dw} are of order -1, k_{Lq} of zero order, and k_{Dq} negligible: κ, τ, s' have the magnitudes of § 7.

We get, as a first approximation,

$$q_0\frac{dq_0}{d\theta} = -\sin\theta, \quad -q_0 u_0 = \cos\theta-u_0^2\,, \quad q_0\frac{dq_0}{d\theta} = -(\gamma^2\tau)\,u_0\!\left(\kappa w_2+\frac{s'}{\gamma^2}q_0\right). \tag{10·3}$$

This gives us the phugoid equations (7·2), with an obvious change in notation, in which u_0 represents the zero-order quantity u', q_0 the zero-order quantity q', and w_2 the second-order quantity w'.

The second approximation gives

$$\left.\begin{aligned}
&q_0\frac{du_1}{d\theta}+q_1\frac{du_0}{d\theta} = -u_0^2, \quad -q_0 u_1-q_1 u_0 = -2u_0 u_1-(\gamma k_{Lw})\,u_0 w_2,\\
&q_0\frac{dq_1}{d\theta}+q_1\frac{dq_0}{d\theta} = -(\gamma^2\tau)\left\{\kappa(u_0 w_3+u_1 w_2)+\frac{s'}{\gamma^2}(u_0 q_1+u_1 q_0)\right\}.
\end{aligned}\right\} \tag{10·4}$$

These equations enable us to find u_1, q_1, w_3.

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In the second approximation we are using

$$\text{(i)}\ \ X = -\rho S k_D u^2, \quad \text{(ii)}\ \ Z = -\rho S k_L u^2\left(1 + k_{Lw}\frac{w}{u}\right), \tag{10·5}$$

so that *the drag is included to the first order in* γ; *the effect of the* w *component of velocity is taken into account in the lift*; and *no assumption is made about the moment of inertia*, except that involved in τ_N being taken of order -2, or larger, and this is generally true for a conventional aeroplane. Hence the second approximation gives a reasonably accurate description of the motion.

This second approximation (without and with screw thrust) has been successfully investigated by Dr G. S. Atkinson of Leeds University, now at the Northampton Polytechnic, London, and the results obtained give interesting information about looping motion, with considerable accuracy. It is hoped to publish the results in the near future. They possess the advantage over the method of step-by-step integration, that the results are not merely numerical, for given numerical data, but are general in terms of functions involving the initial conditions.

Higher approximations are possible in the same way, and, of course, the method can be applied to any first approximation solution.

(II) Standard diving condition: *diving phugoids*

11. In the standard diving condition of the machine γ is $\frac{1}{2}\pi$, s' is the reciprocal of several hundreds; $\cos\gamma\, k_{Lw}$ is a number like 80 or 100, $\cos\gamma\, k_{Lq}$ like 15 or 20, while k_{Dw} and k_{Dq} can be ignored. The equations (3·1) now become

$$\left.\begin{aligned} q'\left(\frac{du'}{d\theta} + w'\right) &= -\sin\theta - u'^2, \\ q'\left(\frac{dw'}{d\theta} - u'\right) &= \cos\theta - u'w' - \cos\gamma\, k_{Lw} u'w' - \cos\gamma\, k_{Lq} s'u'q', \\ q'\frac{dq'}{d\theta} &= -\tau u'(\kappa w' + s'q'). \end{aligned}\right\} \tag{11·1}$$

We know that $q' \equiv (V_{-\gamma}/g)\, q$, and now $V_{-\gamma}/g$ is a rather large number, like 15 or 20, for any conventional aeroplane. *If we use* $g/V_{-\gamma}$ *as defining first-order smallness*, the value of τ_D can be assumed to be of order -3 and s' of order 2.

In practice, we can assume κ to be fairly small (whether positive or negative); we therefore define a problem, to which the method of approximation can be applied, as follows:

Let $\dot\theta$ be so small that q' is of zero order. The third equation (11·1) gives

$$\kappa w' + s'q' = -\frac{1}{\tau_D u'}\, q'\frac{dq'}{d\theta}. \tag{11·2}$$

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The right-hand side of (11·2) is of the third order; hence $\kappa w'$ must be of the second order (since $s'q'$ is of the second order). If then κ is rather small and of order $(g/V_{-\gamma})^{\frac{1}{3}}$, w' is of order $(g/V_{-\gamma})^{\frac{2}{3}}$. The number $\cos\gamma\, k_{Lw}$, which is like 80 or 100, is of order $(g/V_{-\gamma})^{-\frac{2}{3}}$, and so the equations (11·1) become, as a first approximation,

$$q'\frac{du'}{d\theta} = -\sin\theta - u'^2, \quad -q'u' = \cos\theta - (\cos\gamma\, k_{Lw})\, u'w', \quad w' = -\frac{s'}{\kappa}q'. \tag{11·3}$$

In these "diving" phugoids, the air-thrust components are

$$\text{(i)}\ \ X = -\rho S k_D u^2, \quad \text{(ii)}\ \ Z = -\rho S k_L u^2\left(k_{Lw}\frac{w}{u}\right), \tag{11·4}$$

so that *the first approximation is comparatively close to reality*, *the only assumption being* what Lanchester calls *neglecting the moment of inertia*.

Using w' from the third equation (11·3) in the second, the system reduces to

$$q'\frac{du'}{d\theta} = -\sin\theta - u'^2, \quad -\left(1+\frac{s'}{\kappa}\cos\gamma\, k_{Lw}\right)q'u' = \cos\theta, \tag{11·5}$$

which, on eliminating q', give

$$\cos\theta\frac{du'}{d\theta} = \left(1+\frac{s'}{\kappa}\cos\gamma\, k_{Lw}\right)u'(\sin\theta + u'^2). \tag{11·6}$$

We must note one exception, viz. $\kappa = -s'\cos\gamma\, k_{Lw}$, when the second equation (11·5) gives $\cos\theta = 0$. Using $\theta = -\frac{1}{2}\pi$, the first equation (11·5) means, since $q'd/d\theta = (V_{-\gamma}/g)\, d/dt$,

$$\frac{du}{dt} = g\left(1-\frac{u^2}{V_{-\gamma}^2}\right), \tag{11·7}$$

which is of course "parachute" diving motion with $V_{-\gamma}$ as terminal velocity. We have again a singular solution (see remark to equations (8·10)).

If we write, as in (8·3),

$$\frac{\kappa}{s'\cos\gamma\, k_{Lw}} \equiv \frac{K}{1-K}, \tag{11·8}$$

equation (11·6) transforms into

$$\frac{K}{2}\frac{d}{d\theta}\left(\frac{1}{u'^2}\right)\cos\theta + \frac{\sin\theta}{u'^2} = -1,$$

i.e.

$$\frac{d}{d\theta}\left(\frac{\sec^{2/K}\theta}{u'^2}\right) = -\frac{2}{K}\sec^{2/K+1}\theta,$$

which, on integration, gives

$$\frac{\sec^{2/K}\theta}{u'^2} = A - \frac{2}{K}\int\sec^{2/K+1}\theta\, d\theta, \tag{11·9}$$

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where A is an arbitrary constant. $K = \infty$, the exceptional case already mentioned, can be deduced as the limit obtained when in (11·9) we make K tend to ∞.

As an illustration take the case $K = 2$ so that $\kappa = -2s' \cos\gamma\, k_{Lw}$. Write $\tan\theta_0$ for A, and equation (11·9) becomes

$$u'^2 = \frac{\cos\theta_0}{\sin(\theta_0 - \theta)}, \tag{11·10}$$

where θ_0 is an arbitrary angle. We also get by the second equation (11·5) that

$$q' = -\frac{2\cos\theta}{u'}. \tag{11·11}$$

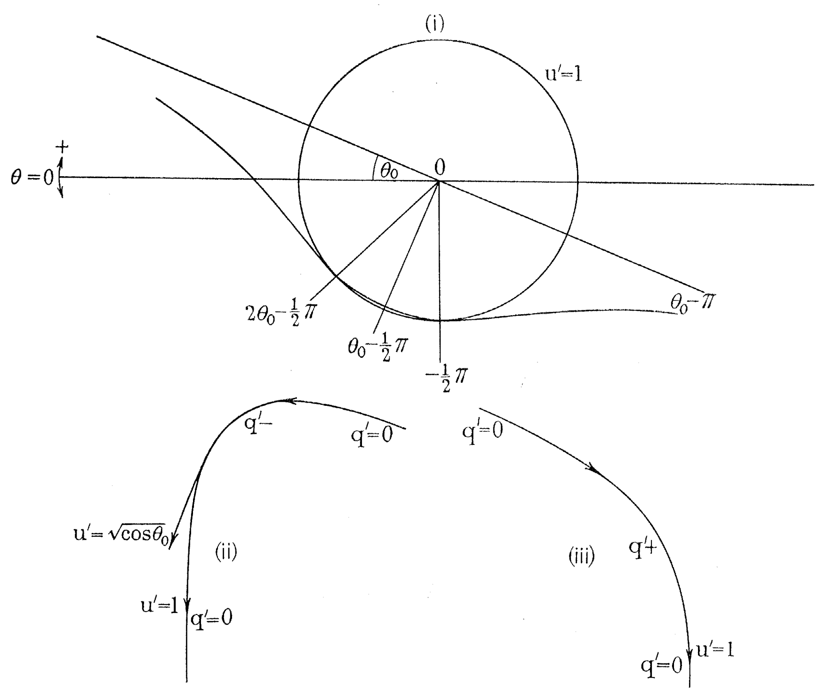

FIG. 4

Further, we get for the radius of curvature

$$\rho = -\frac{1}{2}\frac{V_{-\gamma}^2}{g}\frac{\cos\theta_0}{\cos\theta\sin(\theta_0-\theta)}. \tag{11·12}$$

If we plot u' against θ, as in fig. 4 (i), where u' is the radius vector measured from the pole O, and θ is measured from the original line $\theta = 0$, then we get the hodograph, drawn for the case θ_0 between 0 and $\frac{1}{2}\pi$.

The part of the hodograph from $\theta = \theta_0$ till $\theta = -\frac{1}{2}\pi$ gives a path like that in fig. 4 (ii).

The part of the hodograph from $\theta = -\frac{1}{2}\pi$ till $\theta = \theta_0 - \pi$ gives a path as shown in fig. 4 (iii).

When θ_0 is between $\frac{1}{2}\pi$ and π, or negative, we readily get similar results; the path is always like fig. 4 (ii) or (iii).

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Elevator in rotation during motion

Flattening out from a dive

12. The problem of the dive raises another question of great importance, namely, the process of flattening out from a dive by the use of the elevator. We have here the problem of the motion of an aeroplane with elevator in rotation. This problem can also be dealt with by the method of this paper.

When the elevator is rotated slowly by the pilot during the motion, the effect is that the "condition" of the machine is being changed. Ignoring lag in the effectiveness of the elevator, we can say that, for any instantaneous position of the elevator, there are instantaneous positions of the directions $L = 0$, $M = 0$ in the longitudinal plane, and this gives an instantaneous value of the incidence $i_{-\gamma}$ for the steady glide appropriate to the position of the elevator. We therefore get instantaneous values of γ, $V_{-\gamma}$, s', κ, λ, τ, k_{Lw}, k_{Dw}, k_{Lq}, k_{Dq}, all of which vary, in general, with $i_{-\gamma}$.

Since the axis Gx, which we take to be along the direction $M = 0$, changes in the machine as the elevator is turned, the inclination θ of this axis to the horizontal no longer represents the direction of the body of the machine relative to the air. Let us then take some axis through G fixed in the machine, e.g. in order to illustrate the method roughly, let us neglect the slight change, due to the moving elevator, in the direction of zero lift from which we measure i. Then the orientation of the machine can be represented by $\theta + i_{-\gamma}$ and its angular velocity by $d(\theta + i_{-\gamma})/dt$. We shall, for algebraic convenience, continue to use the symbols q for $\dot{\theta}$, and q' for $(V_{-\gamma}/g)\,q$, although, of course, q is no longer the angular velocity of the machine; it will be convenient to use also

$$\frac{V_{-\gamma}}{g}\frac{d}{dt}(\theta + i_{-\gamma}) \equiv q''. \tag{12·1}$$

Since
$$u = u'V_{-\gamma}, \quad w = w'V_{-\gamma},$$

we find
$$\frac{du}{dt} = gq'\frac{du'}{d\theta} + u'\frac{dV_{-\gamma}}{di_{-\gamma}}\frac{di_{-\gamma}}{dt}, \quad \frac{dw}{dt} = gq'\frac{dw'}{d\theta} + w'\frac{dV_{-\gamma}}{di_{-\gamma}}\frac{di_{-\gamma}}{dt}. \tag{12·2}$$

If η represents the angle that the elevator makes with some standard position, measured positive in the sense $z \to x$, we can write

$$\frac{di_{-\gamma}}{dt} = \frac{di_{-\gamma}}{d\eta}\frac{d\eta}{dt}, \tag{12·3}$$

so that

$$\frac{du}{dt} = g\left\{q'\frac{du'}{d\theta} + u'\frac{d}{di_{-\gamma}}\left(\frac{V_{-\gamma}}{g}\right)\frac{di_{-\gamma}}{d\eta}\frac{d\eta}{dt}\right\}, \quad \frac{dw}{dt} = g\left\{q'\frac{dw'}{d\theta} + w'\frac{d}{di_{-\gamma}}\left(\frac{V_{-\gamma}}{g}\right)\frac{di_{-\gamma}}{d\eta}\frac{d\eta}{dt}\right\}. \tag{12·4}$$

Hence the first two equations (3·1) become

$$\left.\begin{aligned} q'\left(\frac{du'}{d\theta} + w'\right) + u'\frac{d}{di_{-\gamma}}\left(\frac{V_{-\gamma}}{g}\right)\frac{di_{-\gamma}}{d\eta}\frac{d\eta}{dt} &= -\sin\theta - \sin\gamma(u'^2 + k_{Dw}u'w' + k_{Dq}s'u'q'') + \cos\gamma\, u'w', \\ q'\left(\frac{dw'}{d\theta} - u'\right) + w'\frac{d}{di_{-\gamma}}\left(\frac{V_{-\gamma}}{g}\right)\frac{di_{-\gamma}}{d\eta}\frac{d\eta}{dt} &= \cos\theta - \cos\gamma(u'^2 + k_{Lw}u'w' + k_{Lq}s'u'q'') - \sin\gamma\, u'w'. \end{aligned}\right\} \tag{12·5}$$

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With regard to the third equation (3·1), we must go back to the third of the primitive equations (2·2), and rewrite it in the form

$$B\frac{d^2}{dt^2}(\theta+i_{-\gamma}) = M. \tag{12·6}$$

Now
$$\frac{d}{dt}\frac{d}{dt}(\theta+i_{-\gamma}) = \frac{d}{dt}\left(\frac{g}{V_{-\gamma}}q''\right) = \left(\frac{g}{V_{-\gamma}}\right)^2\left\{q'\frac{dq''}{d\theta}-q''\frac{d}{di_{-\gamma}}\left(\frac{V_{-\gamma}}{g}\right)\frac{di_{-\gamma}}{d\eta}\frac{d\eta}{dt}\right\}. \tag{12·7}$$

Hence the third equation (3·1) is replaced by

$$q'\frac{dq''}{d\theta}-q''\frac{d}{di_{-\gamma}}\left(\frac{V_{-\gamma}}{g}\right)\frac{di_{-\gamma}}{d\eta}\frac{d\eta}{dt} = -\tau u'(\kappa w'+s'q''). \tag{12·8}$$

In equations (12·5) and (12·8), γ, $V_{-\gamma}$, κ, λ, τ, s', etc., are all functions of $i_{-\gamma}$; and also by (12·1) we have

$$q'' = q'+\frac{V_{-\gamma}}{g}\frac{di_{-\gamma}}{d\eta}\frac{d\eta}{dt}. \tag{12·9}$$

The variations of γ, $V_{-\gamma}$, κ, etc., are complicated, but they present no insuperable difficulty, especially if the angle through which the elevator is turned does not involve large changes in $i_{-\gamma}$. If this is the case, it is also fairly safe to make $di_{-\gamma}/d\eta$ a constant.

Finally we have to consider $d\eta/dt$. If the elevator is turned at a constant rate there is no difficulty. If not, we can use

$$\frac{d\eta}{dt} = \frac{g}{V_{-\gamma}}q'\frac{d\eta}{d\theta}, \tag{12·10}$$

and any assumed relation between η and θ will, when the problem is solved, become some relation between η and t.

This method has been used successfully by Mr J. Seddon of Leeds University to deal with the flattening out from a dive. Results of considerable interest have been obtained, agreeing with observed facts. It is hoped to publish them soon.

Lanchester's phugoids corrected for drag

13. The neglect of the drag in the phugoids produces the bizarre effect that we can have a horizontal path, or that, in general, the machine continually rises to the same level, although there is no engine to supply the energy dissipated by the air resistance. It is for some purposes (e.g. in dealing with the flattening out from a dive) useful to correct Lanchester's phugoids for the drag, not a complete second approximation, but better than the first.

Let us take equations (3·1) once again. As in § 7, taking standard normal condition, with κ of zero order, and using the orders of τ_N, etc., there assumed, the third equation makes w' a second order quantity in terms of $\sin\gamma$. But let us now write the first two equations correct to the first order in $\sin\gamma$. We get

$$q'\frac{du'}{d\theta} = -\sin\theta-\sin\gamma\,u'^2, \quad -q'u' = \cos\theta-\cos\gamma\,u'^2; \tag{13·1}$$

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We retain the form $\cos\gamma$ (instead of using 1) in the second equation for reasons of mathematical symmetry.

Equations (13·1) define "corrected" Lanchester phugoids. They can be derived also for the standard stalled condition if we assume κ to be fairly large, as is usually the case in this condition. They apply to all conditions of the machine with γ not too large, and κ not too small, so that w' is of the second order.

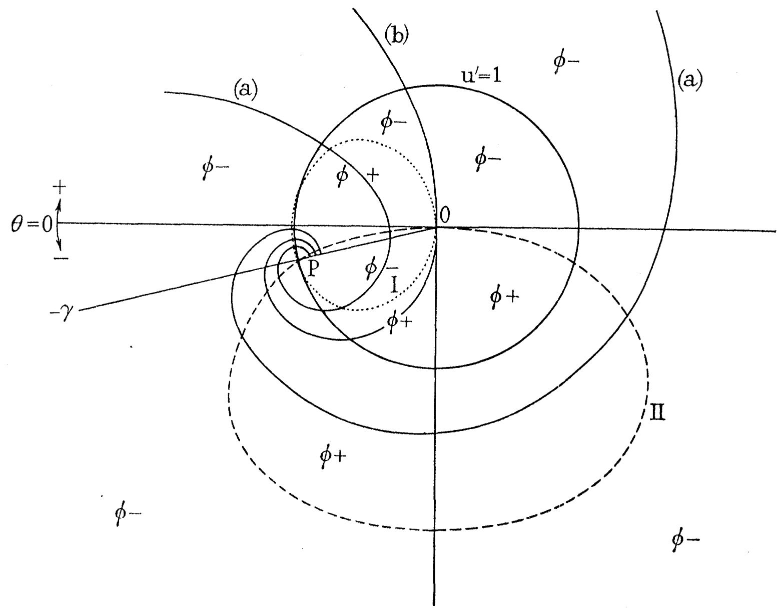

FIG. 5

Eliminating q' between the two equations (13·1), we get

$$(\cos\theta - \cos\gamma\, u'^2)\frac{du'}{d\theta} = u'(\sin\theta + \sin\gamma\, u'^2), \tag{13·2}$$

an obvious extension of (7·4). The radius of curvature is

$$\rho = \frac{V_{-\gamma}^2}{g}\frac{u'^2}{\cos\gamma\, u'^2 - \cos\theta}. \tag{13·3}$$

The equation (13·2) can be readily dealt with if $\sin\gamma \to 0$, when we obtain the Lanchester phugoids in the standard normal condition, or if $\sin\gamma \to 1$ (which can be shown to be applicable to the first approximation in the case of a parachute under suitable conditions). In general, however, when γ is neither small nor nearly $\frac{1}{2}\pi$ a simple integration process does not exist for this equation.

But consider the hodographs defined by the differential equation (13·2). In fig. 5 we have u' as radius vector, measured from the pole O, and θ measured from the original line $\theta = 0$.

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If ϕ is the angle between the radius vector, in the positive sense, and the direction of the tangent, in the sense of θ increasing, we have

$$\tan\phi = \frac{\cos\theta - \cos\gamma\, u'^2}{\sin\theta + \sin\gamma\, u'^2}. \tag{13·4}$$

Plot the curves: I, $u'^2 = \cos\theta/\cos\gamma$; II, $u'^2 = -\sin\theta/\sin\gamma$. I (dotted) is the locus of points on the family of hodographs represented by (13·2) at which the tangents pass through the pole O; II (dashed) is the locus of points at which the tangents are perpendicular to the radius vector from O. The two curves I, II cut at the point P, whose co-ordinates are

$$u' = 1, \quad \theta = -\gamma.$$

The point P, by itself, is the hodograph of the steady glide with gliding angle γ; this is the corrected form of Lanchester's straight line phugoid. The point P is a singular point in the system of curves represented by the differential equation (13·2).

From (13·4) it follows that $\tan\phi$ is positive and negative, respectively, as indicated in the spaces in fig. 5, and an examination of the diagram shows that P is the pole of a family of spirals, each of which surrounds P, while it comes in closer and closer to P, reaching it after an infinite number of revolutions. Further, from (13·1) we see that the sign of q' is the same as the sign of $\cos\gamma\, u'^2 - \cos\theta$, since u' is always positive; hence q' is positive outside I and negative inside I.

Mathematically speaking, each hodograph (*a*) in fig. 5, commencing with some large value of u', goes round the pole O a number of times, and then falls short of O, and describes a spiral inwards round P. Hence each path consists of a number of loops, followed by a number of undulations of diminishing angular amplitude. In fact the corrected phugoid partakes of the character of both loops and undulations, the undulations consisting of a closer and closer approximation to the steady glide in the direction γ. Whether loops will appear or not, in fact, depends on whether the initial point is taken before or after the last loop in the path.

There is one exceptional hodograph, (*b*) in fig. 5, which passes through the pole O. It clearly does so tangentially to the direction $\theta = \pm\frac{1}{2}\pi$, and then curls in towards P. This particular hodograph gives a path, which, after describing loops, describes its last loop (or its first undulation) in the degenerate form of a cusp which points vertically upwards; after this it undulates with diminishing angular amplitude, and approximates more and more closely to the steady glide $\theta = -\gamma$. As in Lanchester's semicircular phugoid, the cusp bears little relation to the truth, since the approximation involves that u' never becomes zero, which it does at a cusp.

We have a descriptive account of the corrected phugoids. More definite results have been obtained by Mr C. P. O'Dowd of Leeds University, who has dealt also with engines in action.

It is easily proved that if in normal condition κ is of the first order and τ of order -3, we get a factor $1/K$ on the left-hand side of the second equation (13·1), where K is as defined by (8·3). The consequent modifications in the results are obvious.

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II. LONGITUDINAL MOTION WITH ENGINES IN ACTION

Equations of motion: first approximation

14. We now proceed to the motion of an aeroplane with engines in action. The aeroplane is still taken to be symmetrical, and the motion to be in the plane of symmetry or longitudinal plane, which is supposed to be fixed and vertical (relative to the air). For simplicity we shall suppose (fig. 6) that the resultant screw thrust is T and acts through the centre of gravity G. Let the x axis make an angle β with the direction of T, Gx being as before that direction in the machine, motion along which, without rotation, gives no moment due to air resistance, the tail plane and elevator being in given and fixed positions. It is not difficult to deal with the more general case where T does not pass through G, but in that case we must take into account the moment due to T. We shall ignore the effects of T on the directions $L = 0$, $M = 0$; they are quite small.

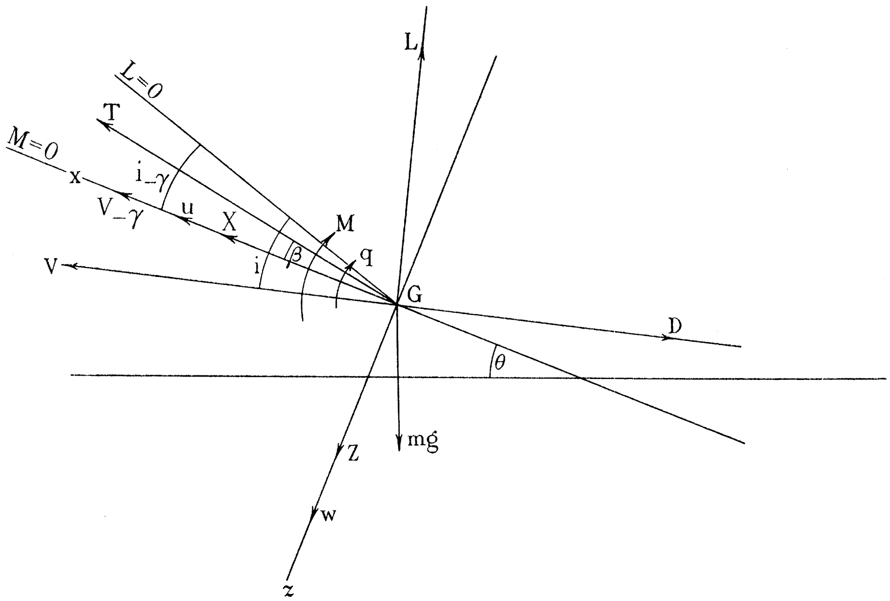

Fig. 6

For any given position of the elevator and a given adjustment of the engines, there is an appropriate steady motion. Let it be a steady climb at angle θ_0, with velocity V_{θ_0}. Then in this steady motion

$$u = V_{\theta_0}, \quad w = 0, \quad \theta = \theta_0, \quad q = 0, \quad T = T_{\theta_0},$$

with the conditions of steady motion

$$0 = -mg\sin\theta_0 + X_{\theta_0} + T_{\theta_0}\cos\beta, \quad 0 = mg\cos\theta_0 + Z_{\theta_0} - T_{\theta_0}\sin\beta, \quad 0 = M_{\theta_0}. \quad (14{\cdot}1)$$

If γ is the gliding angle for the given condition of the machine, then, ignoring viscosity, elasticity, and the effect of T on the air resistance forces we have

$$\frac{X_{\theta_0}}{X_{-\gamma}} = \frac{Z_{\theta_0}}{Z_{-\gamma}} = \left(\frac{V_{\theta_0}}{V_{-\gamma}}\right)^2,$$

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and the conditions for steady motion give

$$\left(\frac{V_{\theta_0}}{V_{-\gamma}}\right)^2 = \frac{\cos(\beta+\theta_0)}{\cos(\gamma-\beta)}, \quad T_{\theta_0} = mg\frac{\sin(\gamma+\theta_0)}{\cos(\gamma-\beta)}. \tag{14·2}$$

For the screw thrust let us assume that, during the flight, the numbers of revolutions and the engine torques are constant; then T is, in accordance with Bairstow's theory, a function of the velocity component along the direction of T. This gives

$$T = T_{\theta_0} f\left(\frac{u\cos\beta - w\sin\beta}{V_{\theta_0}\cos\beta}\right) = mg\frac{\sin(\gamma+\theta_0)}{\cos(\gamma-\beta)} f\left(\frac{u}{V_{\theta_0}} - \frac{w}{V_{\theta_0}}\tan\beta\right), \tag{14·3}$$

where f is some functional form such that $f(1) = 1$.

Writing

$$\frac{V}{V_{-\gamma}} \equiv V', \quad \frac{V_{\theta_0}}{V_{-\gamma}} \equiv V'_{\theta_0}, \quad \frac{u}{V_{-\gamma}} \equiv u', \quad \frac{w}{V_{-\gamma}} \equiv w', \quad \frac{V_{-\gamma}}{g} q \equiv q', \quad \frac{sg}{V^2_{-\gamma}} \equiv s',$$

the equations of motion (2·10) with the screw-thrust included are

$$\left.\begin{aligned} q'\left(\frac{du'}{d\theta}+w'\right) &= -\sin\theta - \sin\gamma V'^2\left\{\frac{u'}{V'}\frac{k_D\left(\frac{w'}{V'}, \frac{s'q'}{V'}\right)}{k_D(0,0)} - \frac{w'}{V'}\cot\gamma\frac{k_L\left(\frac{w'}{V'}, \frac{s'q'}{V'}\right)}{k_L(0,0)}\right\} \\ &\quad + \frac{\sin(\gamma+\theta_0)\cos\beta}{\cos(\gamma-\beta)} f\left(\frac{u'}{V'_{\theta_0}} - \frac{w'}{V'_{\theta_0}}\tan\beta\right), \\ q'\left(\frac{dw'}{d\theta}-u'\right) &= \cos\theta - \cos\gamma V'^2\left\{\frac{w'}{V'}\tan\gamma\frac{k_D\left(\frac{w'}{V'}, \frac{s'q'}{V'}\right)}{k_D(0,0)} + \frac{u'}{V'}\frac{k_L\left(\frac{w'}{V'}, \frac{s'q'}{V'}\right)}{k_L(0,0)}\right\} \\ &\quad - \frac{\sin(\gamma+\theta_0)\sin\beta}{\cos(\gamma-\beta)} f\left(\frac{u'}{V'_{\theta_0}} - \frac{w'}{V'_{\theta_0}}\tan\beta\right), \\ q'\frac{dq'}{d\theta} &= -\tau V'^2\left\{\kappa\frac{w'}{V'} + \frac{s'q'}{V'} + \lambda\frac{s'q'}{V'^2}\frac{dw'}{d\theta} + k_{M2}\left(\frac{w'}{V'}, \frac{s'q'}{V'}\right)\right\}. \end{aligned}\right\} \tag{14·4}$$

It is safe to assume that τ is not radically affected by the state of the engines, so that τ_N, τ_D and τ_S can be taken to be of the orders of magnitude already used above. Using equations (14·4) with the approximations of § 3 we get, instead of (3·1), the approximate equations

$$\left.\begin{aligned} q'\left(\frac{du'}{d\theta}+w'\right) &= -\sin\theta - \sin\gamma(u'^2 + k_{Dw}u'w' + k_{Dq}s'u'q') \\ &\quad + \cos\gamma\, u'w' + \frac{\sin(\gamma+\theta_0)\cos\beta}{\cos(\gamma-\beta)}\left\{f\left(\frac{u'}{V'_{\theta_0}}\right) - \frac{w'}{V'_{\theta_0}}\tan\beta f'\left(\frac{u'}{V'_{\theta_0}}\right)\right\}, \\ q'\left(\frac{dw'}{d\theta}-u'\right) &= \cos\theta - \cos\gamma(u'^2 + k_{Lw}u'w' + k_{Lq}s'u'q') \\ &\quad - \sin\gamma\, u'w' - \frac{\sin(\gamma+\theta_0)\sin\beta}{\cos(\gamma-\beta)}\left\{f\left(\frac{u'}{V'_{\theta_0}}\right) - \frac{w'}{V'_{\theta_0}}\tan\beta f'\left(\frac{u'}{V'_{\theta_0}}\right)\right\}, \\ q'\frac{dq'}{d\theta} &= -\tau u'(\kappa w' + s'q'). \end{aligned}\right\} \tag{14·5}$$

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The quantities k_{Lw}, k_{Dw}, k_{Lq}, k_{Dq} do depend somewhat on θ_0, but it is safe to ignore this circumstance in dealing with approximate solutions of the equations of motion based upon comparison of orders of magnitude.

(I) Standard normal condition: moderate and large power

15. We again use $\sin\gamma$ as the first-order small quantity, and, as in § 6, we take s' to be of the second order; k_{Lw}, k_{Dw} of order -1; k_{Lq} of order zero, and k_{Dq} negligible. In the standard normal condition β is very small. Two divisions occur in the treatment:

(IA) Moderate engine power, so that θ_0 is small;

(IB) Large engine power, so that θ_0 is considerable, of course positive.

In each we consider the three subtypes (*a*), (*b*), (*c*) of § 6.

(IA) Moderate power: *Lanchester's, extended and neutral phugoids*

16. (*a*) *κ of zero order, τ_N of order -2, or larger; Lanchester's phugoids.* Assume q small, so that q' is of zero order. The third equation (14·5) again makes w' a second-order quantity. If now we write the first two equations (14·5) with small quantities omitted, we get, as a first approximation,

$$q'\frac{du'}{d\theta} = -\sin\theta, \quad -q'u' = \cos\theta - u'^2, \quad w' = -\frac{s'}{\kappa}q' - \frac{1}{\kappa\tau u'}q'\frac{dq'}{d\theta}, \tag{16·1}$$

identical with equations (7·2), and we get again Lanchester's phugoids.

This apparently paradoxical result is quite intelligible. The action of the screw makes no difference to the first approximation, because when θ_0 is small the screw thrust itself supplies only a first-order quantity in the equations of motion, being only a small fraction of mg. It is thus seen that in Lanchester's phugoids there is no question of the screw thrust just balancing the drag at each instant; it is the smallness of the drag and the screw thrust separately, compared to the lift and weight, that makes Lanchester's approximation possible. It is best when $\theta_0 = 0$.

(*b*) *κ small, of the first order, τ_N of order -3, or larger.* We get the same results as in § 8, *the extended phugoids*.

(*c*) *κ negligible, τ_N of order -2, or larger.* We get the same results as in § 9, *the neutral phugoids*.

(IB) Large power: *power phugoids: zooming*

17. (*a*) *κ of zero order, τ_N of order -2, or larger; Lanchester power phugoids.* Using the fact that w' is a second-order quantity, as given by the third equation (14·5), the first two equations of (14·5) now become, as a first approximation,

$$q'\frac{du'}{d\theta} = -\sin\theta + \sin\theta_0 f\left(\frac{u'}{V'_{\theta_0}}\right), \quad -q'u' = \cos\theta - u'^2. \tag{17·1}$$

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The first of the equations (17·1) differs from the first of equations (7·2) in the fact that the former contains a term involving f, due to the engines.

Eliminating q' between the two equations (17·1) we get

$$\cos\theta\frac{du'}{d\theta}-u'^2\frac{du'}{d\theta}=\sin\theta\,u'-\sin\theta_0\,u'f\left(\frac{u'}{V'_{\theta_0}}\right),$$

or

$$\frac{1}{u'}\frac{du'}{d\theta}=\frac{\sin\theta-\sin\theta_0 f(u'/V'_{\theta_0})}{\cos\theta-u'^2}. \tag{17·2}$$

We no longer have the Lanchester phugoids. The paths defined by (17·2) are in a sense corrected phugoids, but the correction made here is that account is taken of the considerable air-screw thrust due to θ_0 not being small. We can therefore appropriately describe them as "power" phugoids.

The actual paths of the power phugoids depend, of course, on the nature of the function f. It can be readily seen, by considering the hodographs, that the path consists, in general, of loops followed by undulations of diminishing angular amplitude, tending toward the rectilinear steady motion $u' = V'_{\theta_0}$, $\theta = \theta_0$.

Lanchester's straight line becomes the rectilinear steady motion $u' = V'_{\theta_0}$, $\theta = \theta_0$.

(*b*) *κ of the first order, τ_N of order* -3, *or larger; extended power phugoids.* The equations (14·5) become, with θ_0 considerable,

$$q'\frac{du'}{d\theta}=-\sin\theta+\sin\theta_0 f\left(\frac{u'}{V'_{\theta_0}}\right),\quad -q'u'=\cos\theta-u'^2-k_{Lw}u'w',\quad w'=-\frac{s'}{\kappa}q', \tag{17·3}$$

w' being again, as in § 8, of the first order in $\sin\gamma$.

Using the third equation in the second, we get

$$q'\frac{du'}{d\theta}=-\sin\theta+\sin\theta_0 f\left(\frac{u'}{V'_{\theta_0}}\right),\quad -\left(1+\frac{s'}{\kappa}k_{Lw}\right)q'u'=\cos\theta-u'^2, \tag{17·4}$$

so that, defining K as in (8·3), we have the relation between u' and θ

$$\frac{K}{u'}\frac{du'}{d\theta}=\frac{\sin\theta-\sin\theta_0 f(u'/V'_{\theta_0})}{\cos\theta-u'^2}, \tag{17·5}$$

which can be discussed graphically, or numerially, for any given form f. This has been worked out by Mr O'Dowd, using the form of f given in (17·8) with $\sigma=2$.

(*c*) *κ negligible, τ_N of order* -2, *or larger; neutral power phugoids; zooming.* As in § 9, we can take w' to be of the first order, and write the third equation (14·5) in the same form as (9·1). The equations (14·5) give in fact

$$q'\frac{du'}{d\theta}=-\sin\theta+\sin\theta_0 f\left(\frac{u'}{V'_{\theta_0}}\right),\quad -q'u'=\cos\theta-u'^2-k_{Lw}u'w',\quad q'\frac{dq'}{d\theta}=-\tau s'u'q'; \tag{17·6}$$

and the nature of the approximation is the same as in § 9. Once again $q'=0$ is a first approximation.

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(i) If $d\theta/dt$ is zero initially, then it is zero permanently, so that the machine does not turn, and Gx points in the constant direction θ.

Using (5·1), the first two equations (17·6) give approximately

$$\frac{V_{-\gamma}}{g}\frac{du'}{dt} = -\sin\theta + \sin\theta_0 f\left(\frac{u'}{V'_{\theta_0}}\right), \quad w' = \frac{\cos\theta - u'^2}{u'k_{Lw}}. \tag{17·7}$$

Remembering that θ is constant, the first equation can be satisfied by taking u' constant, say n, so chosen that $f(n/V'_{\theta_0}) = \sin\theta/\sin\theta_0$. Then w' is also constant, and the path is approximately a straight line inclined below θ by the small angle $(\cos\theta - n^2)/n^2k_{Lw}$, and described with uniform speed n.

If $\theta = \theta_0$, then $f(n/V'_{\theta_0}) = 1$, and $n = V'_{\theta_0}$. It follows from (14·2) that, approximately, $n = \surd(\cos\theta_0)$, so that the machine climbs in the direction θ_0 with constant speed given by $u' = \surd(\cos\theta_0)$.

If u' is not constant, the "neutral power" phugoids then obtained will depend on the exact form of the function f.

There is reason to believe that $f(u'/V'_{\theta_0})$ can be written in the form

$$1 + \nu - \nu(u'/V'_{\theta_0})^{\sigma}, \tag{17·8}$$

where ν, σ are positive constants. Then u' satisfies the equation

$$\frac{V_{-\gamma}}{g}\frac{du'}{dt} = \{(1+\nu)\sin\theta_0 - \sin\theta\} - \nu\sin\theta_0\left(\frac{u'}{V'_{\theta_0}}\right)^{\sigma}. \tag{17·9}$$

If $\theta < \sin^{-1}(1+\nu)\sin\theta_0$, we have a terminal value of u' given by

$$u' = V'_{\theta_0}\left(\frac{\overline{1+\nu}\sin\theta_0 - \sin\theta}{\nu\sin\theta_0}\right)^{1/\sigma}; \tag{17·10}$$

and we get, terminally, the straight-line motion just described. This motion can be described as "zooming", since θ_0 is large and positive.

If $\theta > \sin^{-1}(1+\nu)\sin\theta_0$, the machine loses speed continually, and the straight-line terminal motion is not possible.

The motion with u' not constant is, in general, rather complicated. With values of σ like 1 or 2, the motion can be calculated out, if required, without excessive trouble.

(ii) If $d\theta/dt$ is not zero initially, so that it is not zero permanently, we readily find for q' the equation

$$q'\frac{d^2q'}{d\theta^2} = \tau s'\left\{\sin\theta - \sin\theta_0 f\left(-\frac{dq'}{d\theta}\Big/\tau s' V'_{\theta_0}\right)\right\}, \tag{17·11}$$

an obvious extension of (9·9).

III. THREE DIMENSIONAL MOTION

Symmetrical aeroplane

Equations of motion: first approximation

18. Perhaps the most interesting applications of the method of this paper are to the study of approximate solutions of the three-dimensional general motion of an aeroplane.

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We shall take here a glider, or an aeroplane with no screw thrust; the results can be extended to deal with engines in action.

We shall begin with the machine symmetrical, i.e. the rudder and ailerons in neutral positions.

We use the axes Gx, Gz as already defined, and add Gy perpendicular to both, its positive sense being to the right when looking along Gx, so that we have a right-handed set of axes.

Let u, v, w be the components of velocity of G along these axes, and p, q, r the components of angular velocity of the machine about any instantaneous position of the axes. If ψ, θ, ϕ are the usual Eulerian angles appropriate to this problem, namely the angles of yaw, pitch and roll respectively, then we have

$$p = \dot{\phi} - \sin\theta\,\dot{\psi}, \quad q = \cos\phi\,\dot{\theta} + \cos\theta\sin\phi\,\dot{\psi}, \quad r = -\sin\phi\,\dot{\theta} + \cos\theta\cos\phi\,\dot{\psi}. \qquad (18\cdot1)$$

Let h_1, h_2, h_3 be the components of angular momentum about the axes, and A, B, C, D, E, F the usual moments and products of inertia. Then $D = 0$, $F = 0$, and we have

$$h_1 = Ap - Er, \quad h_2 = Bq, \quad h_3 = Cr - Ep. \qquad (18\cdot2)$$

Using X, Y, Z; L, M, N, for the components of force and moment due to the air resistances, the equations of motion are

$$\left.\begin{aligned} m(\dot{u} - rv + qw) &= -mg\sin\theta + X, \\ m(\dot{v} - pw + ru) &= mg\cos\theta\sin\phi + Y, \\ m(\dot{w} - qu + pv) &= mg\cos\theta\cos\phi + Z; \\ \dot{h}_1 - rh_2 + qh_3 &= L, \\ \dot{h}_2 - ph_3 + rh_1 &= M, \\ \dot{h}_3 - qh_1 + ph_2 &= N. \end{aligned}\right\} \qquad (18\cdot3)$$

We now use θ or ϕ or ψ as a new independent variable instead of the time, the choice being dictated by convenience, as we shall soon see. In the longitudinal motion, of course, there was no other choice than θ.

If we choose θ as the new independent variable instead of the time, let us write

$$p \equiv \dot{\theta}P, \quad q \equiv \dot{\theta}Q, \quad r \equiv \dot{\theta}R, \qquad (18\cdot4)$$

so that

$$P = \frac{d\phi}{d\theta} - \sin\theta\frac{d\psi}{d\theta}, \quad Q = \cos\phi + \cos\theta\sin\phi\frac{d\psi}{d\theta}, \quad R = -\sin\phi + \cos\theta\cos\phi\frac{d\psi}{d\theta}. \qquad (18\cdot5)$$

Introduce the notation

$$\frac{V}{V_{-\gamma}} \equiv V', \quad \frac{u}{V_{-\gamma}} \equiv u', \quad \frac{v}{V_{-\gamma}} \equiv v', \quad \frac{w}{V_{-\gamma}} \equiv w', \quad \frac{V_{-\gamma}}{g}\dot{\theta} \equiv q', \quad \frac{sg}{V_{-\gamma}^2} \equiv s', \qquad (18\cdot6)$$

where $V_{-\gamma}$ is the longitudinal gliding velocity of the machine in its given condition, γ the gliding angle.

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The equations of motion (18·3) now become

$$\left.\begin{aligned}
&q'\left(\frac{du'}{d\theta}-Rv'+Qw'\right)=-\sin\theta+X/mg,\\
&q'\left(\frac{dv'}{d\theta}-Pw'+Ru'\right)=\cos\theta\sin\phi+Y/mg,\\
&q'\left(\frac{dw'}{d\theta}-Qu'+Pv'\right)=\cos\theta\cos\phi+Z/mg;\\
&q'\frac{d}{d\theta}\left\{q'\left(P-\frac{E}{A}R\right)\right\}+q'^2\left(-\frac{B-C}{A}QR-\frac{E}{A}PQ\right)=\frac{L}{A}\left(\frac{V_{-\gamma}}{g}\right)^2,\\
&q'\frac{d}{d\theta}\{q'Q\}+q'^2\left(-\frac{C-A}{B}RP+\frac{E}{B}\overline{P^2-R^2}\right)=\frac{M}{B}\left(\frac{V_{-\gamma}}{g}\right)^2,\\
&q'\frac{d}{d\theta}\left\{q'\left(R-\frac{E}{C}P\right)\right\}+q'^2\left(-\frac{A-B}{C}PQ+\frac{E}{C}QR\right)=\frac{N}{C}\left(\frac{V_{-\gamma}}{g}\right)^2.
\end{aligned}\right\}\qquad(18\cdot7)$$

By the theory of dimensions, and ignoring viscosity and elasticity, we can assume that $\quad X/\rho V^2S,\quad Y/\rho V^2S,\quad Z/\rho V^2S;\quad L/\rho V^2Ss,\quad M/\rho V^2Ss,\quad N/\rho V^2Ss,$

where we again take the semi-span s as the characteristic length defining the size of the machine, are functions of the dimensionless arguments

$$\frac{v}{V},\quad \frac{w}{V},\quad \frac{sp}{V},\quad \frac{sq}{V},\quad \frac{sr}{V},$$

or

$$\frac{v'}{V'},\quad \frac{w'}{V'},\quad \frac{s'q'}{V'}P,\quad \frac{s'q'}{V'}Q,\quad \frac{s'q'}{V'}R,\qquad(18\cdot8)$$

where v', w', q', s' are defined by (18·6), and P, Q, R by (18·5).

Let us, in order to get a first approximation, assume that these arguments are small. Since the machine is assumed symmetrical about the longitudinal plane, $X=X_{-\gamma}$, $Z=Z_{-\gamma}$, while Y, L, M, N are zero, in the steady glide. It will be convenient to use altogether Jones's notation (p. 133); after a little manipulation, the equations of motion (18·7) become

$$\left.\begin{aligned}
&q'\left(\frac{du'}{d\theta}-Rv'+Qw'\right)=-\sin\theta-\sin\gamma\,u'^2\quad-\left(\frac{x_w}{k_R}u'w'+\frac{x_q}{k_R}s'u'q'Q\right),\\
&q'\left(\frac{dv'}{d\theta}-Pw'+Ru'\right)=\cos\theta\sin\phi\quad-\left(\frac{y_v}{k_R}u'v'+\frac{y_p}{k_R}s'u'q'P+\frac{y_r}{k_R}s'u'q'R\right),\\
&q'\left(\frac{dw'}{d\theta}-Qu'+Pv'\right)=\cos\theta\cos\phi-\cos\gamma\,u'^2-\left(\frac{z_w}{k_R}u'w'+\frac{z_q}{k_R}s'u'q'Q\right);\\
&q'\frac{d}{d\theta}\left\{q'\left(P-\frac{E}{A}R\right)\right\}+q'^2\left(-\frac{B-C}{A}QR-\frac{E}{A}PQ\right)=-\tau\left(\frac{l_v}{m_q}u'v'+\frac{l_p}{m_q}s'u'q'P+\frac{l_r}{m_q}s'u'q'R\right),\\
&q'\frac{d}{d\theta}\{q'Q\}\quad+q'^2\left(-\frac{C-A}{B}RP+\frac{E}{B}\overline{P^2-R^2}\right)=-\tau\left(\frac{m_w}{m_q}u'w'+s'u'q'Q\right),\\
&q'\frac{d}{d\theta}\left\{q'\left(R-\frac{E}{C}P\right)\right\}+q'^2\left(-\frac{A-B}{C}PQ+\frac{E}{C}QR\right)=-\tau\left(\frac{n_v}{m_q}u'v'+\frac{n_p}{m_q}s'u'q'P+\frac{n_r}{m_q}s'u'q'R\right),
\end{aligned}\right\}$$

(18·9)

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using the definition of τ in (4·7). As in § 3, the values of x_q, z_q used by Jones must be multiplied by c/s, and m_w by s/c; all the other symbols remain unchanged.

In two-dimensional motion $Q = 1$, $v' = P = R = \phi = \psi = 0$; cf. (3·3) using κ from (4·5).

If we take ϕ as the independent variable, we write

$$p \equiv \dot{\phi} P, \quad q \equiv \dot{\phi} Q, \quad r \equiv \dot{\phi} R, \tag{18·10}$$

so that

$$P = 1 - \sin\theta \frac{d\psi}{d\phi}, \quad Q = \cos\phi \frac{d\theta}{d\phi} + \cos\theta \sin\phi \frac{d\psi}{d\phi}, \quad R = -\sin\phi \frac{d\theta}{d\phi} + \cos\theta\cos\phi \frac{d\psi}{d\phi}. \tag{18·11}$$

We now use the notation
$$\frac{V_{-\gamma}}{g} \dot{\phi} \equiv p', \tag{18·12}$$

and instead of (18·9) we have the equations of motion

$$\left.\begin{aligned}
&p'\left(\frac{du'}{d\phi} - Rv' + Qw'\right) = -\sin\theta - \sin\gamma\, u'^2 \quad -\left(\frac{x_w}{k_R} u'w' + \frac{x_q}{k_R} s'u'p'Q\right),\\
&p'\left(\frac{dv'}{d\phi} - Pw' + Ru'\right) = \cos\theta\sin\phi \quad -\left(\frac{y_v}{k_R} u'v' + \frac{y_p}{k_R} s'u'p'P + \frac{y_r}{k_R} s'u'p'R\right),\\
&p'\left(\frac{dw'}{d\phi} - Qu' + Pv'\right) = \cos\theta\cos\phi - \cos\gamma\, u'^2 - \left(\frac{z_w}{k_R} u'w' + \frac{z_q}{k_R} s'u'p'Q\right);\\
&p'\frac{d}{d\phi}\left\{p'\left(P - \frac{E}{A}R\right)\right\} + p'^2\left(-\frac{B-C}{A} QR - \frac{E}{A} PQ\right) = -\tau\left(\frac{l_v}{m_q} u'v' + \frac{l_p}{m_q} s'u'p'P + \frac{l_r}{m_q} s'u'p'R\right),\\
&p'\frac{d}{d\phi}\{p'Q\} \quad + p'^2\left(-\frac{C-A}{B} RP + \frac{E}{B}\overline{P^2 - R^2}\right) = -\tau\left(\frac{m_w}{m_q} u'w' + s'u'p'Q\right),\\
&p'\frac{d}{d\phi}\left\{p'\left(R - \frac{E}{C}P\right)\right\} + p'^2\left(-\frac{A-B}{C} PQ + \frac{E}{C} QR\right) = -\tau\left(\frac{n_v}{m_q} u'v' + \frac{n_p}{m_q} s'u'p'P + \frac{n_r}{m_q} s'u'p'R\right).
\end{aligned}\right\} \tag{18·13}$$

There is no harm done in using the same letters P, Q, R for the definitions (18·5) and for the definitions (18·11), as it is clear that (18·5) must be used when θ is the independent variable, and (18·11) must be used when ϕ is the independent variable. We now turn to ψ as independent variable, and we have a third set of definitions for P, Q, R.

If we take ψ as the independent variable, we write

$$p \equiv \dot{\psi} P, \quad q \equiv \dot{\psi} Q, \quad r \equiv \dot{\psi} R, \tag{18·14}$$

so that

$$P = \frac{d\phi}{d\psi} - \sin\theta, \quad Q = \cos\phi \frac{d\theta}{d\psi} + \cos\theta\sin\phi, \quad R = -\sin\phi \frac{d\theta}{d\psi} + \cos\theta\cos\phi. \tag{18·15}$$

We use the notation
$$\frac{V_{-\gamma}}{g} \dot{\psi} \equiv r', \tag{18·16}$$

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and instead of (18·9), (18·13) we have the equations of motion

$$\left.\begin{aligned}
r'\left(\frac{du'}{d\psi}-Rv'+Qw'\right) &= -\sin\theta-\sin\gamma\, u'^2 -\left(\frac{x_w}{k_R}u'w'+\frac{x_q}{k_R}s'u'r'Q\right),\\
r'\left(\frac{dv'}{d\psi}-Pw'+Ru'\right) &= \cos\theta\sin\phi -\left(\frac{y_v}{k_R}u'v'+\frac{y_p}{k_R}s'u'r'P+\frac{y_r}{k_R}s'u'r'R\right),\\
r'\left(\frac{dw'}{d\psi}-Qu'+Pv'\right) &= \cos\theta\cos\phi-\cos\gamma\, u'^2-\left(\frac{z_w}{k_R}u'w'+\frac{z_q}{k_R}s'u'r'Q\right);\\
r'\frac{d}{d\psi}\left\{r'\left(P-\frac{E}{A}R\right)\right\}+r'^2\left(-\frac{B-C}{A}QR-\frac{E}{A}PQ\right) &= -\tau\left(\frac{l_v}{m_q}u'v'+\frac{l_p}{m_q}s'u'r'P+\frac{l_r}{m_q}s'u'r'R\right),\\
r'\frac{d}{d\psi}\{r'Q\} +r'^2\left(-\frac{C-A}{B}RP+\frac{E}{B}\overline{P^2-R^2}\right) &= -\tau\left(\frac{m_w}{m_q}u'w'+s'u'r'Q\right),\\
r'\frac{d}{d\psi}\left\{r'\left(R-\frac{E}{C}P\right)\right\}+r'^2\left(-\frac{A-B}{C}PQ+\frac{E}{C}QR\right) &= -\tau\left(\frac{n_v}{m_q}u'v'+\frac{n_p}{m_q}s'u'r'P+\frac{n_r}{m_q}s'u'r'R\right).
\end{aligned}\right\} \tag{18·17}$$

The sets of equations of motion (18·9), (18·13) and (18·17) are really equivalent to one another. It is useful to note how τ occurs in all the equations for the angular momentum.

(I) Standard normal condition: three subtypes

19. If the aeroplane is in standard normal condition, we use once again $\sin\gamma$ as the standard of smallness, and distinguish the cases in accordance with the value of κ or m_w/m_q, as in § 6.

Let us assume v', w' small, and the rotation of the machine to be small, so that p', q', r' are all of zero order. This means that if we use θ as independent variable, with the equations of motion (18·9) then q' is of zero order, and the corresponding P, Q, R of (18·5) are never large; if we use ϕ in (18·13), then p' is of zero order, and P, Q, R of (18·11) are never large; and if we use ψ in (18·17), then r' is of zero order and P, Q, R of (18·15) are never large. We shall take s' to be of the second order.

20. (*a*) *κ of zero order; τ_N of order -2, or larger; three-dimensional phugoids. Immelmann turn.* Let us use θ as independent variable in equations of motion (18·9).

The fifth equation gives at once that w' is of the second order. Noting that x_q, y_p, y_r are usually negligible; and taking into account the fact that usually, for normal condition, x_w/k_R, y_v/k_R are of zero order, while z_w/k_R, z_q/k_R are of order -1; then since v' is small, while w' is of the second order, we at once deduce that the first three equations of (18·9) give

$$q'\frac{du'}{d\theta}=-\sin\theta,\quad q'Ru'=\cos\theta\sin\phi,\quad -q'Qu'=\cos\theta\cos\phi-u'^2. \tag{20·1}$$

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To the order of our first approximation, we are in (18·3) putting

$$\text{(i)}\ X = 0, \quad \text{(ii)}\ Y = 0, \quad \text{(iii)}\ Z = -\rho S k_L u^2. \tag{20·2}$$

Thus (i) *we neglect the drag*. (If the air screws are in action and the thrust is small compared to the weight, we also neglect the thrust, and get the same result—as happened in the longitudinal problem.) Also

(ii) *we neglect the effect of the side-slip on the air resistance;* and

(iii) *we assume "constant incidence", neglecting the effect on the lift of changes in the w velocity component.*

Finally, we assume that *the machine adjusts itself in the Lanchester manner about each of the three axes*. This is perhaps more difficult to imagine in lateral than in longitudinal motion, but as basis for a first approximation has the same validity in both.

The fourth and sixth equations of (18·9) give that

$$\left.\begin{aligned} l_v v' + l_p s'q'P + l_r s'q'R &= 0, \\ \text{and} \qquad n_v v' + n_p s'q'P + n_r s'q'R &= 0, \end{aligned}\right\} \tag{20·3}$$

to the first order. Now it is a well-known fact that, of the six rotary derivatives contained in (20·3), l_p stands out as predominantly large. In fact l_p is a quantity of order -1, while of the other five derivations only l_r is at all large, namely, like 2. We therefore suggest that, in our first approximation, we can take $P = 0$.

This suggestion is much nearer the truth than appears on the surface at first. For if we take the typical values of the rotary derivatives we find that by adding four or five times the second equation (20·3) to the first (20·3) we get $P = 0$ with considerable accuracy.

We therefore obtain the following equations for the first approximation:

$$q'\frac{du'}{d\theta} = -\sin\theta, \quad q'Ru' = \cos\theta\sin\phi, \quad -q'Qu' = \cos\theta\cos\phi - u'^2, \quad P = 0; \tag{20·4}$$

where $$q' \equiv (V_{-\gamma}/g)\,\dot{\theta},$$

and $$P = \frac{d\phi}{d\theta} - \sin\theta\frac{d\psi}{d\theta}, \quad Q = \cos\phi + \cos\theta\sin\phi\frac{d\psi}{d\theta}, \quad R = -\sin\phi + \cos\theta\cos\phi\frac{d\psi}{d\theta}.$$

If we start off with the equations of motion (18·13) using ϕ as independent variable, and argue in the same way as we have just done with the equations of motion (18·9), we find the first approximation:

$$p'\frac{du'}{d\phi} = -\sin\theta, \quad p'Ru' = \cos\theta\sin\phi, \quad -p'Qu' = \cos\theta\cos\phi - u'^2, \quad P = 0; \tag{20·5}$$

where $$p' \equiv (V_{-\gamma}/g)\,\dot{\phi},$$

and $$P = 1 - \sin\theta\frac{d\psi}{d\phi}, \quad Q = \cos\phi\frac{d\theta}{d\phi} + \cos\theta\sin\phi\frac{d\psi}{d\phi}, \quad R = -\sin\phi\frac{d\theta}{d\phi} + \cos\theta\cos\phi\frac{d\psi}{d\phi},$$

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If we start off with the equations of motion (18·17) using ψ as independent variable, we get the first approximation:

$$r'\frac{du'}{d\psi} = -\sin\theta, \quad r'Ru' = \cos\theta\sin\phi, \quad -r'Qu' = \cos\theta\cos\phi - u'^2, \quad P = 0; \qquad (20\cdot6)$$

where $$r' \equiv (V_{-\gamma}/g)\,\dot{\psi},$$

and $$P = \frac{d\phi}{d\psi} - \sin\theta, \quad Q = \cos\phi\frac{d\theta}{d\psi} + \cos\theta\sin\phi, \quad R = -\sin\phi\frac{d\theta}{d\psi} + \cos\theta\cos\phi.$$

It is obvious that the three sets (20·4), (20·5) and (20·6) are identical equations. We have here indicated all three ways of deducing the same result in order to make our process clear.

The equations just obtained, in any of the three forms, define three-dimensional phugoids, extensions into three dimensions of the Lanchester phugoids (7·2), which are the special case in which the motion is longitudinal.

It is quite easy to solve the equations of the three-dimensional phugoids. Let us (since, as will be seen, the rotation ψ in a constant sense is a feature of the motion) use ψ as the independent variable; the equations (20·6) give

$$\left.\begin{aligned} r'\frac{du'}{d\psi} &= -\sin\theta, \quad r'u'\left(-\sin\phi\frac{d\theta}{d\psi} + \cos\theta\cos\phi\right) = \cos\theta\sin\phi, \\ r'u'\left(\cos\phi\frac{d\theta}{d\psi} + \cos\theta\sin\phi\right) &= u'^2 - \cos\theta\cos\phi, \quad \frac{d\phi}{d\psi} = \sin\theta. \end{aligned}\right\} \qquad (20\cdot7)$$

The first and second equations (20·7) give

$$\cos\theta\sin\phi\frac{du'}{d\psi} - u'\sin\theta\sin\phi\frac{d\theta}{d\psi} + u'\sin\theta\cos\theta\cos\phi = 0; \qquad (20\cdot8)$$

and if we write $d\phi/d\psi$ for $\sin\theta$ in the last term in (20·8) we get

$$\cos\theta\sin\phi\frac{du'}{d\psi} - u'\sin\theta\sin\phi\frac{d\theta}{d\psi} + u'\cos\theta\cos\phi\frac{d\phi}{d\psi} = 0,$$

so that we deduce at once the important result

$$u'\cos\theta\sin\phi = a, \qquad (20\cdot9)$$

where a is an arbitrary constant. We get from the second and third equations of (20·7) that

$$r'\cos\theta = u'\sin\phi, \qquad (20\cdot10)$$

so that we deduce $$\frac{V_{-\gamma}}{g}\dot{\psi} \equiv r' = \frac{a}{\cos^2\theta}. \qquad (20\cdot11)$$

We also find from the first, second and third equations (20·7) that

$$\cos\theta\frac{du'}{d\psi} - u'\sin\theta\frac{d\theta}{d\psi} = u'^2\frac{du'}{d\psi}\cos\phi,$$

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i.e. $$\frac{d}{du'}(u'\cos\theta) = u'^2\cos\phi, \tag{20·12}$$

while, by (20·9), $$\sin\phi = \frac{a}{u'\cos\theta}, \quad \cos\phi = \frac{\surd(u'^2\cos^2\theta - a^2)}{u'\cos\theta}.$$

Equation (20·12) becomes

$$\frac{d}{du'}(u'\cos\theta) = u'^2\frac{\surd(u'^2\cos^2\theta - a^2)}{u'\cos\theta},$$

which gives $$\surd(u'^2\cos^2\theta - a^2) = \tfrac{1}{3}u'^3 + A,$$

or $$u'^2\cos^2\theta = a^2 + (\tfrac{1}{3}u'^3 + A)^2, \tag{20·13}$$

where A is an arbitrary constant, analogous to A in (7·5) for Lanchester's phugoids.

Thus we have the results

$$u'\cos\theta\sin\phi = a, \quad u'\cos\theta\cos\phi = \tfrac{1}{3}u'^3 + A, \quad \tan\phi = \frac{a}{\tfrac{1}{3}u'^3 + A}, \quad \dot{\psi} = \frac{ag}{V_{-\gamma}\cos^2\theta}; \tag{20·14}$$

which lead to a relation between θ and ϕ, namely

$$a\cot\phi = \tfrac{1}{3}a^3\sec^3\theta\,\mathrm{cosec}^3\phi + A. \tag{20·15}$$

The special case $a = 0$ gives, of course, the Lanchester two-dimensional phugoids.

The three-dimensional phugoids can be explained very briefly as follows:

If for simplicity we take the case $A = 0$ (which, when $a = 0$ gives the semicircular phugoid) we have from (20·13)

$$u'^2\cos^2\theta = a^2 + \tfrac{1}{9}u'^6. \tag{20·16}$$

When $\cos^2\theta = 1$, we have $u'^2 = a^2 + \frac{1}{9}u'^6$; we get two equal values of u'^2 from this cubic for u'^2 if $a^2 = 2/\surd 3$, and these equal values are $u'^2 = \surd 3$ (the third root gives u'^2 negative and can therefore be discarded). We find that $\cos^2\theta$ is a minimum for this value of u'^2, and that this minimum is unity. Hence we must have θ permanently 0 (or π), and we have circular motion of the aeroplane. The banking angle ϕ is given by $\tan\phi = \surd 2$; $\dot{\psi}$ is equal to the constant $ag/V_{-\gamma}$, and the radius of the circular motion is the constant $nV_{-\gamma}^2/ag$ (where n is the value of u' in this motion), so that, since $n^2 = \surd 3$ and $a^2 = 2/\surd 3$, the radius is $\surd\frac{3}{2}V_{-\gamma}^2/g$.

The equation (20·15) for ϕ in terms of θ gives, with $\cos\theta = 1$, the relation

$$a\cot\phi = \tfrac{1}{3}a^3\,\mathrm{cosec}^3\phi + A.$$

Putting $A = 0$, we get for ϕ the equation

$$\cos\phi - \cos^3\phi = \tfrac{1}{3}a^2. \tag{20·17}$$

If $a^2 < 2/\surd 3$, (20·17) gives two different real values of ϕ;

if $a^2 = 2/\surd 3$, (20·17) gives two equal real values of ϕ;

if $a^2 > 2/\surd 3$, (20·17) gives no real values of ϕ.

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Hence we must have $a^2 \leqslant 2/\surd 3$. If $a^2 = 2/\surd 3$, the motion is circular and steady; if $a^2 < 2/\surd 3$, the motion is more general and variable.

Take, for example, $a^2 = 10/9$; then $\cos\theta = 1$ gives $u_1'^2 = 2$ and $u_2'^2 = \surd 6 - 1$; when $u_1'^2 = 2$, $\tan\phi_1 = \frac{1}{2}\surd 5$; and when $u_2'^2 = \surd 6 - 1$, $\tan\phi_2 = \frac{1}{5}\surd 2(\surd 6+1)^{\frac{3}{2}}$.

The nature of the motion is now clear. Starting at a lowest point, where $u_1'^2 = 2$, the aeroplane climbs with θ increasing from zero to a maximum, and then decreasing again to zero, when the aeroplane is at a highest point with velocity $u_2'^2 = \surd 6 - 1$. It then descends to a second lowest position at the same level as the first lowest; climbs again to a second highest position, at the same level as the first highest; and so on. The non-dimensional velocity, u', decreases from $\surd 2$ to $\surd(\surd 6 - 1)$ during each climb, and increases from $\surd(\surd 6 - 1)$ to $\surd 2$ during each descent. All the highest positions are at the same level; all the lowest positions are at the same level.

Meanwhile the vertical plane containing the x axis rotates about the vertical, always in the same sense, with angular velocity $\dot\psi = (ag/V_{-\gamma})\sec^2\theta$. At the same time, the aeroplane rotates about the x axis, as is indicated by the variation in ϕ. At a lowest position, $\tan\phi_1 = \frac{1}{2}\surd 5$; at a highest position $\tan\phi_2 = \frac{1}{5}\surd 2(\surd 6+1)^{\frac{3}{2}}$; and ϕ oscillates between its smallest value ϕ_1 at a lowest position and its greatest value ϕ_2 at a highest position.

More generally, with A still zero, let us put

$$a^2 = 3\alpha(1-\alpha)(2-\alpha). \tag{20·18}$$

We find that when $\cos^2\theta = 1$, we have the values of u'^2 given by

$$u_1'^2 = 3(1-\alpha), \quad u_2'^2 = -\tfrac{3}{2}(1-\alpha) + \tfrac{3}{2}\surd(1+6\alpha-3\alpha^2). \tag{20·19}$$

Hence α must lie between 0 and 1.

The maximum value of a^2 is easily seen to be given by $\alpha = 1 - 1/\surd 3$, and this maximum is $2/\surd 3$, i.e. the case of circular motion. The range from $\alpha = 1 - 1/\surd 3$ to $\alpha = 1$ gives the same pair of values $u_1'^2$, $u_2'^2$, but interchanged, as the range from $\alpha = 0$ to $\alpha = 1 - 1/\surd 3$. Thus, the case $a^2 = 10/9$ that we have just discussed is given by $\alpha = \frac{1}{3}$, so that $u_1'^2 = 2$, $u_2'^2 = \surd 6 - 1$; and by $\alpha = \frac{1}{3}(4 - \surd 6)$, so that $u_1'^2 = \surd 6 - 1$, $u_2'^2 = 2$.

It is interesting to find the angle by which ψ increases between a lowest and a highest position. We find from (20·7) and (20·11) that

$$\frac{d\psi}{du'} = -\frac{a}{\sin\theta\cos^2\theta}, \tag{20·20}$$

and this gives for the "apsidal angle" between u_1' and u_2' the value

$$27a\int_{u_2'}^{u_1'} \frac{u'^3\,du'}{(u'^6 + 9a^2)\surd(9u'^2 - u'^6 - 9a^2)}, \tag{20·21}$$

if α is so chosen that $u_1' > u_2'$: we can make this choice without any restriction on the result by taking α between 0 and $1 - 1/\surd 3$.

It can be easily shown that when a^2 has its maximum value the apsidal angle is $\frac{1}{2}\surd 2\,\pi$. Also it follows from the work of Mr D. Temple Roberts of Leeds University,

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that as a^2 diminishes the apsidal angle diminishes, and that when $a^2 \to 0$, which also means $\alpha \to 0$, the apsidal angle $\to \frac{1}{2}\pi$, so that in going from a lowest position to a highest position and then back to a lowest position, ψ increases by π.

When $u_1'^2 = 3(1-\alpha)$ we have, with $\cos\theta = 1$,

$$\sin^2\phi = \alpha(2-\alpha).$$

Hence $\sin\phi = \surd(2\alpha-\alpha^2)$, so that when α is small, ϕ is practically zero at a lowest point of the path. When $u_2'^2 = -\frac{3}{2}(1-\alpha)+\frac{3}{2}\surd(1+6\alpha-3\alpha^2)$, we have, with $\cos\theta = 1$,

$$\sin^2\phi = \tfrac{1}{2}(1-\alpha)\{(1-\alpha)+\surd(1+6\alpha-3\alpha^2)\},$$

and when α is small this gives $\sin\phi = 1-2\alpha^2$,

so that ϕ is practically $\frac{1}{2}\pi$ at a highest point of the path.

We have then the Immelmann turn. Mr Roberts has calculated the motion explicitly, and the analogy to the Immelmann turn is quite close.

There is no difficulty in dealing with $A \neq 0$. It has been worked out in some detail already: the nature of the motion is like that obtained when $A = 0$. The Lanchester phugoids, undulating and looping, are obtained as limiting cases $a \to 0$. It is hoped to publish the work soon.

If the engines are in action with moderate power, we get the same three-dimensional phugoids.

21. (*b*) *κ small, of the first order; τ_N of order -3, or larger; extended three-dimensional phugoids.* As in § 8, and using ψ as independent variable, we now have from the fifth equation (18·17)

$$w' = -\frac{s'}{\kappa} r'Q, \tag{21·1}$$

so that the third equation now becomes, approximately,

$$-r'Qu' = \cos\theta\cos\phi - u'^2 + \frac{s'}{\kappa}\frac{z_w}{k_R} r'Qu'. \tag{21·2}$$

The fourth and sixth equations (18·17) give again $P = 0$, and so we get the paths defined by

$$r'\frac{du'}{d\psi} = -\sin\phi, \quad r'Ru' = \cos\theta\sin\phi, \quad -\left(1+\frac{s'}{\kappa}\frac{z_w}{k_R}\right) r'Qu' = \cos\theta\cos\phi - u'^2, \quad P = 0; \tag{21·3}$$

where

$$r' \equiv \frac{V_{-\gamma}}{g}\dot\psi, \text{ and } P = \frac{d\phi}{d\psi} - \sin\theta, \quad Q = \cos\phi\frac{d\theta}{d\psi} + \cos\theta\sin\phi, \quad R = -\sin\phi\frac{d\theta}{d\psi} + \cos\theta\cos\phi.$$

Let us write K as in (8·3).

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Equations (21·3) become, after a little manipulation,

$$\left.\begin{aligned} r'\frac{du'}{d\psi} &= -\sin\theta, \quad r'u'\left(-\sin\phi\frac{d\theta}{d\psi}+\cos\theta\cos\phi\right) = \cos\theta\sin\phi, \\ r'u'\frac{d\theta}{d\psi} &= Ku'^2\cos\phi-\cos\theta+(1-K)\cos\theta\cos^2\phi, \quad \frac{d\phi}{d\psi} = \sin\theta. \end{aligned}\right\} \tag{21·4}$$

The first, second and fourth equations (21·4) are identical with the first, second and fourth equations (20·7). Hence we at once deduce that

$$u'\cos\theta\sin\phi = a. \tag{21·5}$$

We get from the second and third equations (21·4) that

$$r'u'\cos\theta = Ku'^2\sin\phi+(1-K)\cos\theta\sin\phi\cos\phi,$$

so that we deduce $$\frac{V_{-\gamma}}{g}\dot\psi \equiv r' = K\frac{a}{\cos^2\theta}+\frac{1-K}{a}\cos\theta\sin^2\phi\cos\phi. \tag{21·6}$$

Further, from the first and third equations (21·4) we obtain

$$\cos\theta\frac{du'}{d\psi}-u'\sin\theta\frac{d\theta}{d\psi} = Ku'^2\frac{du'}{d\psi}\cos\phi+(1-K)\frac{du'}{d\psi}\cos\theta\cos^2\phi,$$

i.e. $$\frac{d}{du'}(u'\cos\theta) = Ku'^2\cos\phi+(1-K)\cos\theta\cos^2\phi, \tag{21·7}$$

while by (21·5) $$\sin\phi = \frac{a}{u'\cos\theta}, \quad \cos\phi = \frac{\sqrt{(u'^2\cos^2\theta-a^2)}}{u'\cos\theta}.$$

Equation (21·7) therefore becomes

$$\frac{d}{du'}(u'\cos\theta) = Ku'^2\frac{\sqrt{(u'^2\cos^2\theta-a^2)}}{u'\cos\theta}+(1-K)\cos\theta\frac{u'^2\cos^2\theta-a^2}{u'^2\cos^2\theta},$$

i.e. $$u'\frac{d}{du'}\sqrt{(u'^2\cos^2\theta-a^2)}+(K-1)\sqrt{(u'^2\cos^2\theta-a^2)} = Ku'^3.$$

This can be integrated and we obtain, corresponding to (20·13), the result

$$u'^2\cos^2\theta = a^2+\left(\frac{K}{K+2}u'^3+Au'^{1-K}\right)^2, \tag{21·8}$$

where A is an arbitrary constant.

We therefore have, corresponding to (20·14),

$$\left.\begin{aligned} u'\cos\theta\sin\phi &= a, \quad u'\cos\theta\cos\phi = \frac{K}{K+2}u'^3+Au'^{1-K}, \\ \tan\phi &= \frac{a}{\frac{K}{K+2}u'^3+Au'^{1-K}}, \quad \dot\psi = \frac{ag}{V_{-\gamma}\cos^2\theta}\left\{K+\frac{1-K}{a^2}\cos^3\theta\sin^2\phi\cos\phi\right\}, \end{aligned}\right\} \tag{21·9}$$

and, corresponding to (20·15),

$$a\cot\phi = \frac{K}{K+2}a^3\sec^3\theta\,\mathrm{cosec}^3\phi+Aa^{1-K}\sec^{1-K}\theta\,\mathrm{cosec}^{1-K}\phi. \tag{21·10}$$

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If we let $K \to 1$, so that we really have κ of zero order, we get all the equations of the three-dimensional phugoids discussed in § 20.

Hence we have obtained "extended" three-dimensional phugoids. Moderate engine power does not affect them. These paths are more applicable to actual machines than those of § 20: they are now being studied for κ positive and κ negative.

22. (*c*) κ *negligible*. This case does not, in the three-dimensional problem, appear to yield equations that can be dealt with in a simple manner.

We shall deal with (II) Standard Diving Condition elsewhere.

(III) Standard stalled condition: *the slow spin*

23. In the standard stalled condition, not only is τ much smaller than, and m_q different from, their values in the normal and diving conditions, but also l_p is now practically zero. (If the incidence is beyond the standard stall, the effect of autorotation comes in and l_p becomes negative. We shall in this paper not go beyond the standard stall. The spin with larger incidence is being investigated.)

We shall consider here, in an approximate manner, the case of stalling incidence on the assumption that the stalling is not artificially delayed by such devices as slots, etc., so that in the standard stalled condition γ is only a moderate angle, say 15°. It is therefore possible to make a very rough first approximation in which $\sin\gamma$ is considered small, even if it is as much as $\frac{1}{4}$. This is, of course, not very satisfactory, but at the present stage of our general study of aeroplane motion, this represents, at any rate, a first attack on a difficult subject.

If now we take $\sin\gamma$, a number like $\frac{1}{4}$, to represent first-order smallness, then τ_S is of order -1, and, using the small machine discussed by Jones, we take $\tau s'$ to be somewhat less than unity. We shall use the equations (18·17) with ψ as independent variable. It is, in fact, quite convenient to use θ if we desire to do so, but it seems more reasonable to use ψ in the case of a spin in which once again a feature is the increase of ψ always in the same sense.

At stalling incidence, k_R is a number like 0·6 for conventional machines. Further, x_q, y_p, y_r are always negligible, while z_q is practically zero at the stall. Also, at stalling incidence
$$\frac{x_w}{k_R},\quad \frac{y_v}{k_R},\quad \frac{z_w}{k_R}$$
are all either like unity or considerably less.

It follows, assuming that r' is of zero order, and that v', w' are small, of the first order, at least, that the first three equations of (18·17), can to a first and rough approximation be written
$$r'\frac{du'}{d\psi} = -\sin\theta,\quad r'Ru' = \cos\theta\sin\phi,\quad -r'Qu' = \cos\theta\cos\phi - u'^2. \qquad (23\cdot1)$$
Using the values of Q, R of (18·15), we get the alternative form
$$r'\frac{du'}{d\psi} = -\sin\theta,\quad r'u'\frac{d\theta}{d\psi} = u'^2\cos\phi - \cos\theta,\quad r'\cos\theta = u'\sin\phi. \qquad (23\cdot2)$$

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Note. In the standard normal condition z_w/k_R is a number of order -1, in the first approximation appropriate to that condition. Hence w' must be of the second order, in order that it may be omitted in the third equation (18·17). In the present problem z_w/k_R is of zero order; hence w' need only be of the first order, in order that we may neglect it in the third equation (18·17).

In the last three equations of (18·17), we neglect E/A, E/C, which are quantities of the same order as $\sin\gamma$ in a conventional aeroplane at stalling incidence, as well as $(A-B)/C$, which is also small, usually. Also, at the standard stall l_p, n_p, and n_r are negligible. Hence the last three equations (18·17) become

$$\left.\begin{aligned} r'\frac{d}{d\psi}(r'P)-\frac{B-C}{A}r'^2QR &= -\left(\frac{\tau l_v}{m_q}\right)_{-2}u'v'+\left(\tau s'\frac{-l_r}{m_q}\right)_{-1}r'u'R,\\ r'\frac{d}{d\psi}(r'Q)-\frac{C-A}{B}r'^2RP &= -\left(\frac{\tau m_w}{m_q}\right)_{-2}u'w'-(\tau s')_0\,r'u'Q,\\ r'\frac{d}{d\psi}(r'R) \qquad\qquad &= \left(\tau\frac{-n_v}{m_q}\right)_0 u'v'. \end{aligned}\right\} \tag{23·3}$$

The constants are written in such a way that each quantity enclosed in brackets is a positive number, and the suffix attached to each such pair of brackets indicates the order of magnitude of the quantity inside, in terms of $\sin\gamma$ at standard stalling incidence.

Let us again assume that P, Q, R are no larger than of zero order. Then the third equation (23·3) yields the first approximation

$$\frac{d}{d\psi}(r'R) = 0. \tag{23·4}$$

This means that the angular velocity about the axis Gz is approximately constant, so that we get a spin.

The nature of the approximation is that in (18·3) we make once again

$$\text{(i)}\ X=0,\quad \text{(ii)}\ Y=0,\quad \text{(iii)}\ Z=-\rho Sk_L u^2; \tag{23·5}$$

to which we now add (iv) $N = 0$.

Hence *we are* again *neglecting the effects* (i) *of the drag*, (ii) *of the side-slip*, and (iii) *of changes in the w velocity component*, on the air resistance. *We are* (iv) also *neglecting the moment about the z axis* or axis of yaw. Finally, *we assume that the machine adjusts itself instantaneously about the pitching and rolling axes* in the Lanchester manner.

The result (23·4) enables us to write

$$\frac{V_{-\gamma}}{g}r \equiv r'R = \frac{1}{c}, \tag{23·6}$$

where c is an arbitrary constant. Then we get from the second equation (23·1)

$$u' = c\cos\theta\sin\phi, \tag{23·7}$$

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and from the third equation (23·2)

$$\left.\begin{aligned} r' &= c\sin^2\phi, \\ \text{so that} \qquad \dot{\psi} &= \frac{cg}{V_{-\gamma}}\sin^2\phi. \end{aligned}\right\} \tag{23·8}$$

The radius of curvature of the horizontal projection of the path is

$$\frac{u\cos\theta}{\dot{\psi}} = \frac{V_{-\gamma}u'\cos\theta}{gr'/V_{-\gamma}} = \frac{V^2_{-\gamma}}{g}\frac{u'\cos\theta}{r'}, \tag{23·9}$$

and this is equal to

$$\frac{V^2_{-\gamma}}{g}\frac{\cos^2\theta}{\sin\phi}. \tag{23·10}$$

The first two equations of (23·2) give, with (23·7),

$$cr'\frac{d}{d\psi}(\cos\theta\sin\phi) = -\sin\theta, \quad cr'\frac{d\theta}{d\psi} = \frac{c^2\cos\theta\sin^2\phi\cos\phi - 1}{\sin\phi},$$

so that

$$\cos\theta\cos\phi\frac{d\phi}{d\theta}(1-c^2\cos\theta\sin^2\phi\cos\phi) = \sin\theta\sin\phi(2-c^2\cos\theta\sin^2\phi\cos\phi). \tag{23·11}$$

Equation (23·11) gives the relation between θ and ϕ; and, with (23·7), (23·8), the problem is solved.

The constant c can be defined as follows. At $\theta = 0$, $\cos\theta = 1$, let ϕ have the value ϕ_0, and u' be called n; then

$$c = \frac{n}{\sin\phi_0}. \tag{23·12}$$

The equation (23·11) is somewhat complicated. It can be simplified in appearance by making the transformation

$$\left(\frac{2c^2}{3}\right)^3\cos^3\theta\sin^3\phi \equiv \xi, \quad \left(\frac{2c^2}{3}\right)^4\cos^4\theta\sin^2\phi \equiv \eta; \tag{23·13}$$

when it reduces to

$$\frac{d\eta}{d\xi} = \sqrt{(\eta - \xi^{\frac{4}{3}})}. \tag{23·14}$$

But simple integration does not seem possible. We can, however, make our analysis yield a simple result if we make a restriction on the problem considered. This consists in assuming that the quantity $c^2\cos\theta\sin^2\phi\cos\phi$ remains small. Equation (23·11) now becomes

$$\frac{d}{d\theta}(\cos^2\theta\sin\phi) = 0,$$

or

$$\sin\phi = \frac{b}{\cos^2\theta}, \tag{23·15}$$

where b is another arbitrary constant; in fact $b = \sin\phi_0$.

The velocity at $\theta = 0$ is given by $n = bc$.

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We now get $$u' = \frac{bc}{\cos\theta}, \quad r' = \frac{b^2c}{\cos^4\theta}, \quad \dot{\psi} = \frac{b^2cg}{V_{-\gamma}}\Big/\cos^4\theta. \tag{23·16}$$

Also, we now find that $$\frac{d\theta}{dt} = -\frac{g}{V_{-\gamma}}\frac{\cos^2\theta}{bc},$$

i.e. $$\frac{d(\tan\theta)}{dt} = -\frac{g}{bcV_{-\gamma}},$$

so that $$\tan\theta = -\frac{g}{bcV_{-\gamma}}t, \tag{23·17}$$

t being measured from the time when $\theta = 0$: as time progresses, θ becomes more and more negative.

The radius of curvature of the horizontal projection of the path is, by (23·10),

$$\frac{V_{-\gamma}^2}{bg}\cos^4\theta. \tag{23·18}$$

We can get the approximate distance described by the centre of gravity. For consider σ, where

$$d\sigma \equiv u\,dt; \tag{23·19}$$

it is the integral of the distance as measured along Gx (and since v', w' are small, this is approximately the actual distance described by G). Write

$$\frac{\sigma g}{V_{-\gamma}^2} \equiv \sigma'; \tag{23·20}$$

then $$d\sigma' = \frac{g}{V_{-\gamma}^2}u\,dt = \frac{u'}{r'}d\psi = -\frac{b^2c^2}{\cos^3\theta}d\theta,$$

so that $$\sigma' = -\tfrac{1}{2}b^2c^2\{\sec\theta\tan\theta + \log_e(\sec\theta + \tan\theta)\},$$

and $$\sigma = -\frac{V_{-\gamma}^2}{2g}b^2c^2\{\sec\theta\tan\theta + \log_e(\sec\theta + \tan\theta)\}; \tag{23·21}$$

σ' (and therefore also σ) being measured from the place where $\theta = 0$.

Since, after $t = 0$, θ is actually negative, $\tan\theta$ is negative, and $\log(\sec\theta + \tan\theta)$ is also negative, so that σ', and therefore also σ, is positive, and increases with the time.

We have obtained a rough presentation of the development of an "incipient spin" of moderate but stalled incidence. As time progresses the aeroplane turns downwards, as is indicated by (23·17); its speed increases, as indicated by (23·16); its spin about the vertical increases, as shown by (23·16); and the radius of curvature of the horizontal projection of its path decreases, as shown by (23·18).

The applicability of these results is restricted by the fact that r' is of zero order, that P, Q, R must not become large, and that $c^2\cos\theta\sin^2\phi\cos\phi$ is small compared to unity.

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It can be shown that the restrictions amount to the condition that $\sin\theta$ must be comparatively small, so that the results (23·16)–(23·18) only apply to the beginning of the spin, if, at $\theta = 0$, there exists a rolling displacement ϕ_0, while the rolling rotation then is zero, and the velocity u' is not above a certain limit.

In order to deal with the later development of the spin we must use equation (23·11) without the assumed restriction that $c^2\cos\theta\sin^2\phi\cos\phi$ remains small.

It is satisfactory to note that we can deal, although in a roughly approximate manner, with the initial stage of the spin, by the use of comparatively simple analysis. It is a slow spin, since r' is assumed of zero order, and thus $\dot{\psi}$ is small, of the order $g/V_{-\gamma}$. This means that the spin, if it were fully developed, would be at the rate of three or four turns per minute.

Aeroplane with displaced controls: additional moments

24. We have so far taken the aeroplane to be symmetrical. Let us now suppose the controls displaced and held fixed. Since the symmetry of the aeroplane is only slightly disturbed, the areas of the rudder and ailerons being small compared to the wings, we can use the same non-dimensional derivatives as before, but we must add the effects of the displaced controls.

If the ailerons are turned through an angle ξ radians, starboard (right) down, port (left) up, the elevator through an angle η radians, $z \to x$, and the rudder through an angle ζ radians, $x \to y$, then, ξ, η, ζ being moderate, the additional moments about the axes Gx, Gy, Gz can be written in Jones's notation, suitably extended,

$$\left.\begin{array}{lll} -\rho V^2 Ss\dfrac{A}{ms^2}l_\xi\xi, & -\rho V^2 Ss\dfrac{B}{ms^2}m_\xi\xi, & -\rho V^2 Ss\dfrac{C}{ms^2}n_\xi\xi; \\ -\rho V^2 Ss\dfrac{A}{ms^2}l_\eta\eta, & -\rho V^2 Ss\dfrac{B}{ms^2}m_\eta\eta, & -\rho V^2 Ss\dfrac{C}{ms^2}n_\eta\eta; \\ -\rho V^2 Ss\dfrac{A}{ms^2}l_\zeta\zeta, & -\rho V^2 Ss\dfrac{B}{ms^2}m_\zeta\zeta, & -\rho V^2 Ss\dfrac{C}{ms^2}n_\zeta\zeta. \end{array}\right\} \qquad (24·1)$$

The additional forces along the axes are not required.

We can with moderate incidence ignore m_ξ, l_η, n_η, l_ζ, m_ζ. For n_ζ (see Jones, p. 85) in a conventional machine we can use a number like 0·2 in diving or normal conditions, and a number like 0·1 in stalled condition. With aileron setting not beyond $\pm 20°$, l_ξ is a number like 3 in diving or normal conditions, and like 1·5 in stalled condition; n_ξ is a number like $-0·1$ at diving or normal, and like $-0·15$ at stalled condition.

We need not make any estimate of m_η, since it is only required in order to maintain the rotation about the y axis, and does not, as we shall soon see, affect the first approximation paths.

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If we use θ as independent variable, then in the last three equations of (18·9) we have on the right-hand sides:

$$\left.\begin{aligned}&-\tau\left(\frac{l_v}{m_q}u'v'+\frac{l_p}{m_q}s'u'q'P+\frac{l_r}{m_q}s'u'q'R+\frac{l_\xi}{m_q}u'^2\xi\right),\\&-\tau\left(\frac{m_w}{m_q}u'w'+s'u'q'Q+\frac{m_\eta}{m_q}u'^2\eta\right),\\&-\tau\left(\frac{n_v}{m_q}u'v'+\frac{n_p}{m_q}s'u'q'P+\frac{n_r}{m_q}s'u'q'R+\frac{n_\xi}{m_q}u'^2\xi+\frac{n_\zeta}{m_q}u'^2\zeta\right);\end{aligned}\right\} \tag{24·2}$$

where the non-dimensional u', v', ... are defined with reference to $V_{-\gamma}$ of the aeroplane when $\xi=0$, $\eta=0$, $\zeta=0$; and where P, Q, R are defined as in (18·5). Corresponding values are easily written down for ϕ or for ψ as independent variable.

(I) Standard normal condition

(a) κ of zero order

25. We again have subtypes depending on the value of m_w/m_q or κ; but we shall only deal with κ of zero order. It is easy to extend to small values of κ.

In the standard normal condition, with $\sin\gamma$ defining first-order smallness, τ_N is of order -2 or larger. If κ is of zero order, we get that w' is of the second order. Several cases can be considered; we shall discuss two, which give rise to interesting problems.

26. (a, i) *Small asymmetry: three-dimensional phugoids.* Let ξ be of such a size that $l_\xi\,\xi$ is a quantity of the first order. This makes ξ numerically not more than about 3 or 4°. We see that $n_\xi\xi$ is smaller than of the second order. Then, using (24·2) as in § 20, we can, to the first order, put

$$l_v v'+l_p s'q'P+l_r s'q'R+u'l_\xi\xi=0. \tag{26·1}$$

Again, let ζ be of such a size that $n_\zeta\zeta$ is a quantity of the second order, which means that ζ is numerically not more than about 5 or 6°. We can, to the first order, put

$$n_v v'+n_p s'q'P+n_r s'q'R+u'n_\xi\xi+u'n_\zeta\zeta=0. \tag{26·2}$$

With the usual values of the rotary derivatives at normal incidence we deduce from (26·1) that, to the first order,

$$(l_p s')\,q'P+(l_\xi\xi)\,u'=0, \tag{26·3}$$

so that, as a first approximation,

$$\frac{q'P}{u'}=-\frac{l_\xi\xi}{l_p s'}\equiv-\mu,\quad\text{where}\quad q'=\frac{V_{-\gamma}}{g}\dot{\theta};\quad P=\frac{d\phi}{d\theta}-\sin\theta\frac{d\psi}{d\theta}; \tag{26·4}$$

and μ is at most of zero order value, defined by the comparatively small quantity $l_\xi\xi$.

Since l_r is also quite considerable, like 2, it may be useful to reduce the coefficient of $q'R$ by combining equation (26·1) with some appropriate number, k, times equation (26·2). With Jones's machine, p. 184, we can use $k=4$ or 5.

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If v' and $q'R$ are in this way practically cancelled out of existence, we get that, to the first order,

$$(\overline{l_p+kn_p s'})\,q'P+(\overline{l_\xi+kn_\xi}\xi+kn_\zeta\zeta)\,u'=0; \tag{26·5}$$

and if we write

$$\frac{\overline{l_\xi+kn_\xi}\xi+kn_\zeta\zeta}{\overline{l_p+kn_p s'}}\equiv\mu, \tag{26·6}$$

we have once again equation (26·4) as a first approximation.

With ϕ or ψ as independent variable, we get respectively, instead of (26·4),

$$\frac{p'P}{u'}=-\mu,\quad\text{where}\quad p'=\frac{V_{-\gamma}}{g}\dot{\phi},\quad\text{and}\quad P=1-\sin\theta\frac{d\psi}{d\phi}; \tag{26·7}$$

or

$$\frac{r'P}{u'}=-\mu,\quad\text{where}\quad p'=\frac{V_{-\gamma}}{g}\dot{\psi},\quad\text{and}\quad P=\frac{d\phi}{d\psi}-\sin\theta. \tag{26·8}$$

Equations (26·4), (26·7) and (26·8) are of course the same, and we can use whichever we like, depending on the choice of θ, ϕ or ψ as independent variable.

If $\mu=0$ we get $P=0$ as in § 20, so that the problem there discussed is a special case of what we are now considering. We have $\mu=0$ if $\xi=0$, $\zeta=0$, i.e. if the machine is symmetrical and the asymmetric controls are not displaced from their neutral positions. But we can also get μ zero for non-zero values of ξ, ζ: it will be readily verified that settings in which ξ, ζ satisfy the condition

$$\lambda\xi+\zeta=0, \tag{26·9}$$

will give μ practically zero, λ being a positive number like 2 or 3, its exact value depending on the detailed construction of the machine.

This kind of setting is a "perverse" one, since it makes the ailerons turn the machine in one sense and the rudder in the opposite sense; this is of course the meaning of μ being zero.

If $\mu=0$, we have the three-dimensional phugoids of § 20, defined by equations (20·7) with ψ as independent variable. The approximation means once again that *we neglect the drag, the air resistance effect of side-slip, and the air resistance effect of changes in the w velocity component.* While we assume that the machine adjusts itself in the Lanchester manner about each of the yawing, pitching and rolling axes.

There is an apparent difficulty due to the fact that the machine is not symmetrical when ξ, ζ are not zero, so that its steady glide is in general on a helix. If in this helical glide the banking angle is ϕ_1, then it is easy to prove that the new gliding angle γ_1, the new gliding velocity $u'_1 V_{-\gamma}$, and the rotation $(g/V_{-\gamma})\,r'_1$, are given approximately by

$$\gamma_1=\gamma\sec\phi_1,\quad u_1'^2=\sec\phi_1,\quad r'_1 u'_1=\tan\phi_1. \tag{26·10}$$

But the non-steady motion is again given by (20·4), (20·5) or (20·6); so that *an aeroplane possesses only one set of three-dimensional phugoids for any given condition.*

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With $\mu \neq 0$, the equations (20·7) become

$$r'\frac{du'}{d\psi} = -\sin\theta, \quad r'u'\left(-\sin\phi\frac{d\theta}{d\psi}+\cos\theta\cos\phi\right) = \cos\theta\sin\phi,$$

$$r'u'\left(\cos\phi\frac{d\theta}{d\psi}+\cos\theta\sin\phi\right) = u'^2-\cos\theta\cos\phi, \quad \frac{d\phi}{d\psi}+\mu\frac{u'}{r'} = \sin\theta. \tag{26·11}$$

The approximation has the same significance as when $\mu = 0$.

We get again equations (20·8) and (20·12) namely

$$\cos\theta\sin\phi\frac{du'}{d\psi}-u'\sin\theta\sin\phi\frac{d\theta}{d\psi}+u'\sin\theta\cos\theta\cos\phi = 0, \tag{26·12}$$

$$\frac{d}{du'}(u'\cos\theta) = u'^2\cos\phi. \tag{26·13}$$

In (26·12) we use the value of $\sin\theta$ given by the last equation of (26·11), and we deduce

$$\frac{d}{d\psi}(u'\cos\theta\sin\phi) = -\mu\frac{u'^2}{r'}\cos\theta\cos\phi;$$

so that, by (26·13),
$$\frac{d}{d\psi}(u'\cos\theta\sin\phi) = -\frac{\mu\cos\theta}{r'}\frac{d}{du'}(u'\cos\theta)$$

$$=\mu\cot\theta\frac{d}{d\psi}(u'\cos\theta),$$

using the first equation of (26·11).

We thus find a relation between u', θ, ϕ in the form

$$d(u'\cos\theta\sin\phi) = \mu\cot\theta\, d(u'\cos\theta) \tag{26·13}$$

an obvious extension of (20·9) which is obtained when $\mu = 0$.

We have then the beginning of the study of what we may call the unsymmetrical three-dimensional phugoids.

27. (a, ii) *Large aileron displacement: the slow roll.* Let us now suppose that ξ is so large that $l_\xi\xi$ is a quantity of zero order; this means that ξ is an angle like $\frac{1}{3}$ radian, or 20°. Let ζ be zero, or small as in (a, i). We shall use ϕ as independent variable, and the equations of motion (18·13) with the terms due to the controls included as in (24·2).

Owing to the size of $l_\xi\xi$, we can no longer use the assumptions that p', P, Q, R are of zero order. Let us now assume that p' is large and of order -1, while P is of zero order, and Q, R are small, of the first order. The components u', w' are still assumed small, of orders to be soon defined.

The first equation (18·13) gives at once the approximate result

$$u' = a, \tag{27·1}$$

where a is an arbitrary constant of zeio order.

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If τ_N *is of order* -3, *or larger*, the fifth equation (18·13), using (24·2), makes w' a second-order quantity, since κ or m_w/m_q is of zero order, and also E is small compared to A, B, C.

The fourth equation, using (24·2), gives, taking only the largest quantities on both sides,

$$p'\frac{d}{d\phi}(p'P) = -\frac{a\tau}{m_q}(l_p s'p'P + al_\xi \xi).$$

Writing $p' \equiv (V_{-\gamma}/g)\,\dot{\phi}$, we get, since $P\dot{\phi} = p$, that

$$\frac{dp}{dt} = -\frac{g}{V_{-\gamma}}\frac{a\tau l_p s'}{m_q}\left(p + \frac{ga}{V_{-\gamma}}\frac{l_\xi \xi}{l_p s'}\right), \tag{27·2}$$

and the coefficient outside the bracket on the right-hand side is of order -1. Hence p tends very quickly, in a fraction of a second, to the terminal constant value

$$p = -\frac{ag}{V_{-\gamma}}\frac{l_\xi \xi}{l_p s'}, \quad \text{i.e.} \quad p'P = -a\frac{l_\xi \xi}{l_p s'}. \tag{27·3}$$

Now we have postulated that Q and R are first-order quantities, which is, by (18·11), seen to imply that $d\theta/d\phi$ and $d\psi/d\phi$ are small. Hence to our degree of approximation we can write

$$P = 1, \tag{27·4}$$

and we get

$$p' = -\mu a, \tag{27·5}$$

where μ is once again the constant $l_\xi \xi/l_p s'$, but is now a large quantity, of order -1; p' as postulated is also of order -1, while the rolling velocity p is of zero order, since $p = (g/V_{-\gamma})\,p'$.

The aeroplane therefore rolls approximately at a constant rate, going through a complete revolution in a time like 6 seconds, the slow roll.

From the sixth equation (18·13), using (24·2), we deduce that v' is of the first order, since n_v is in the case of normal condition a comparatively small number. If then we turn to the second and third equations of (18·13) we get, approximately,

$$\mu a\left(\frac{dv'}{d\phi} + aR\right) = -\cos\theta\sin\phi, \quad \mu a(v' - aQ) = a^2 - \cos\theta\cos\phi,$$

so that

$$\frac{d}{d\phi}\left(\frac{v'}{a}\right) + R = -\frac{1}{\mu a^2}\cos\theta\sin\phi, \quad \frac{v'}{a} - Q = \frac{1}{\mu a^2}(a^2 - \cos\theta\cos\phi). \tag{27·6}$$

But let us examine the sixth equation (18·13) in detail, using (24·2). Putting $P = 1$, $p' = -\mu a$, and ignoring E/C, we get, since τ is of order -3, μ of order -1, s' of order 2, R small, Q small, and $n_\xi \xi + n_\zeta \zeta$ also of the second order of small quantities, that this equation reduces to

$$n_v \frac{v'}{a} - n_p s'\mu = 0,$$

assuming that the aeroplane is not directionally neutral and n_v exists. In this case we have, approximately,

$$\frac{v'}{a} = \frac{n_p}{n_v}s'\mu, \tag{27·7}$$

a first order constant quantity in the case of Jones's machine (p. 184).

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Thus equations (27·6) become

$$\mu R = -\frac{\cos\theta\sin\phi}{a^2}, \quad \mu Q = \frac{\cos\theta\cos\phi}{a^2} - \alpha, \tag{27·8}$$

where $\alpha \equiv 1 - \mu^2 s' n_p / n_v$, a quantity of zero order.

We therefore get, by (18·11),

$$\mu a^2\left(\cos\phi\frac{d\theta}{d\phi} + \cos\theta\sin\phi\frac{d\psi}{d\phi}\right) = \cos\theta\cos\phi - \alpha a^2,$$

$$\mu a^2\left(-\sin\phi\frac{d\theta}{d\phi} + \cos\theta\cos\phi\frac{d\psi}{d\phi}\right) = -\cos\theta\sin\phi,$$

so that

$$\frac{d\theta}{d\phi} = \frac{\cos\theta}{\mu a^2} - \frac{\alpha}{\mu}\cos\phi, \quad \cos\theta\frac{d\psi}{d\phi} = -\frac{\alpha}{\mu}\sin\phi. \tag{27·9}$$

If then a is not too large, we get by a graphical examination of equations (27·9) that $d\theta/d\phi$ is always fairly small, being sometimes positive and sometimes negative; so that, beginning with more or less horizontal motion, θ remains small; hence $d\psi/d\phi$ also remains small. It is also found that on the whole, as ϕ increases, θ decreases algebraically, which means that with each complete roll the machine loses some height; this agrees with experience.

If we make θ small in (27·9) we get approximately

$$\mu(\theta - \theta_0) = \frac{\phi}{a^2} - \alpha\sin\phi, \quad \mu(\psi - \psi_0) = -\alpha(1 - \cos\phi), \tag{27·10}$$

where θ_0, ψ_0 as the values of θ, ψ when $\phi = 0$.

We have therefore obtained a first approximation to the slow roll. It is true that this has been done with the assumption that τ_N is of order -3, but, as already pointed out, § 8, this is not a serious restriction.

The significance of the first approximation is that we again *neglect the drag, the effect of side-slip and the effect of changes in the w velocity component. We* now *assume the Lanchester adjustment for the yawing and pitching axes*; about the rolling axis there is such a rate of roll that the rolling moment due to the displaced ailerons balances approximately the air-resistance moment due to the roll.

The slow roll has been worked out in detail by Mr S Kirkby of Leeds University, who has extended the investigation to first order values of κ. He has also dealt with the engines in action, and has examined the aileron roll in a dive, as well as other rolling motions. In so far as the results can be compared with observation, they are satisfactory.

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